Spectral Triples and the Super-Virasoro Algebra

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Abstract

We construct infinite dimensional spectral triples associated with representations of the super-Virasoro algebra. In particular the irreducible, unitary positive energy representation of the Ramond algebra with central charge c and minimal lowest weight $h = c/24$ is graded and gives rise to a net of even θ-summable spectral triples with non-zero Fredholm index. The irreducible unitary positive energy representations of the Neveu-Schwarz algebra give rise to nets of even θ-summable generalised spectral triples where there is no Dirac operator but only a superderivation.

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Contents

1 Introduction

In this paper we make a vital step in the "noncommutative geometrization" program for Conformal Field Theory, that is in the search of noncommutative geometric invariants associated with conformal nets and their representations.

As we are here in the framework of quantum systems with infinitely many degrees of freedom, natural objects to look for are spectral triples in the sense of Connes and Kasparov, see [6]. While there are important situations where these objects enter in Quantum Field Theory, see e.g. [6, 14], the novelty of our work is that our spectral triple depends on the sector with respect to the vacuum representation, according to what was proposed by the QFT index theorem [17].

Let us briefly explain the root of our work. A fundamental object in Connes' Noncommutative Geometry is a spectral triple, a noncommutative extension of the concept of elliptic pseudo-differential operator, say of the Dirac operator. Basically, a (graded) spectral triple $(\mathfrak{A}, \mathcal{H}, Q)$ consists of a $\mathbb{Z}_2$-graded algebra $\mathfrak{A}$ acting on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ and an odd selfadjoint linear operator Q on $\mathcal{H}$, with certain spectral summability properties and bounded graded commutator with elements of $\mathfrak{A}$. A spectral triple gives rise to a cyclic cocycle on A, the Chern character, that evaluates on K_0-theory elements of the even part $\mathfrak{A}_+$ of $\mathfrak{A}$.

In the present infinite-dimensional case, the right summability condition is the trace-class property of the heat kernel, $\mathrm{Tr}(e^{-\beta Q^2}) < \infty$, $\beta > 0$. The involved cohomology is entire cyclic cohomology [6] and the corresponding Chern character is given by the Jaffe-Lesniewski-Osterwalder formula [13] (see also [5, 10]).

Concerning Quantum Field Theory, one expects a natural occurrence of spectral triples in the supersymmetric frame. We recall a related QFT index theorem for certain massive models on the cylinder in the vacuum representation [14].

As explained in [17], one may aim for a QFT index theorem, a noncommutative analog of the Atiyah-Singer index theorem for systems with infinitely many degrees of freedom, where a Doplicher-Haag-Roberts representation (superselection sector, see [12]) represents the analog of an elliptic operator. While the operator algebraic and analytic structure behind the DHR theory is well understood, in particular by Jones theory of subfactors [18], little is known about the possible noncommutative geometrical counterpart.

One would like to get a map

$$\rho \longrightarrow \tau_\rho$$

that associates a noncommutative geometric quantity τ_ρ to a sector ρ.

Now the operator algebraic approach to low-dimensional Conformal Quantum Field Theory (CFT) has shown to be very powerful as can be seen in particular by the classification of chiral CFTs with central charge $c < 1$ [16] and the construction of new models [16, 20]. Therefore, CFT offers a natural framework for the noncommutative geometry set-up.

Namely, we may want to look for a spectral triple associated with a sector in CFT. In order to have such a structure we may further want to restrict our attention to the supersymmetric case, namely to superconformal field theory (SCFT).

The present paper is a first step in this direction by constructing spectral triples associated with (unitary, positive energy) representations of the super-Virasoro algebra [9].

We now explain the actual content of this paper. In a recent paper by three of us [4] we have set up the operator algebraic picture for SCFT. In particular, we have given an interpretation of Neveu-Schwarz and Ramond sectors as representations of a Fermi net on S^1 or of its promotion to the double cover of S^1, respectively.

Starting with the super-Virasoro algebra, we have then defined the super-Virasoro net SVir_c for a given admissible central charge value c, see [9]. Then Neveu-Schwarz and Ramond representations SVir_c correspond to representations of the Neveu-Schwarz algebra and of the Ramond algebra, respectively. As shown in [4], this is at least the case if $c < 3/2$.

In order to have the necessary tools to deal with super-derivations, we provide a quick technical summary in Section 2. Many statements are similar to the case of ungraded derivations, but specialised to our setting. In order to make clear what is meant by spectral triples and why we are interested in them, we state the classical definitions and their extensions to our setting in conformal field theory.

Our main results start in Section 4 with the Ramond algebra. In this case graded representations are supersymmetric inasmuch as the odd element G_0 of the Ramond algebra is a square root of the shifted conformal Hamiltonian $L_0 - c/24$. In the spirit of Algebraic Quantum Field Theory, starting from any such representation, we can define the net of von Neumann algebras generated by the corresponding quantum fields (the Bose and Fermi energy-momentum tensors). If $e^{-\beta(L_0-c/24)}$ is trace class for all $\beta > 0$ we obtain a net of (θ-summable) graded spectral triples by intersecting the local von Neumann algebras with the domain of the superderivation induced by G_0. However in principle such intersections may reduce to the multiples of the identity operator or in any case may be “too small” and this fact gives rise to a highly nontrivial technical problem.

In this paper we show how to solve the above problem and in fact we prove that the algebra of smooth elements for the superderivation intersects every local von Neumann algebra in a weakly dense *-subalgebra. A similar problem has been studied in the free supersymmetric case in [2], where a crucial simplification occurred due to the Weyl commutation relations and the fact that the smeared free Fermi fields are bounded operators.

In particular starting from the irreducible unitary Ramond representation with central charge c and minimal lowest weight $h = c/24$, which is the unique irreducible graded unitary representation of the Ramond algebra with central charge c, we can define a nontrivial net of local even spectral triples.

For the Neveu-Schwarz algebra (in particular the vacuum sector is a representation of this algebra) the structure is definitely less manifest because the odd elements G_r, the Fourier modes of the Fermi stress-energy tensor, are indexed by $r \in \mathbb{Z} + 1/2$, so none of them provides us with a supercharge operator, an odd square root of the conformal Hamiltonian.

In fact no such Dirac type operator can exist in this case. It is however natural to expect that the spectral triples appearing in the Ramond case have a local manifestation also in the Neveu-Schwarz case.

We will indeed generalise the notion of spectral triple to the case where there is no supercharge operator but only a superderivation δ whose square δ^2 is the derivation $[L_0, \cdot]$ implemented by the conformal Hamiltonian. The situation is here even different from the one treated in [15] where a flow on the algebra with a super-KMS functional exists.

Starting with an irreducible unitary positive energy representation of the Neveu-Schwarz algebra we shall construct a net of graded, generalised θ-summable spectral triples associated with the corresponding Neveu-Schwarz net of von Neumann algebras. Here it is interesting to note that, while for the Ramond algebra we get a net of spectral triples on S^1, for the the Neveu-Schwarz algebra the net will live only on the double cover $S^{1(2)}$ because the local superderivations cannot be consistently defined on S^1.

For the Ramond, the JLO cocycles appear and can be investigated. Concerning the Neveu-Schwarz case it is unclear whether a corresponding cyclic cocycle can be directly defined, see Section 5.

As we shortly mention in the outlook, we hope to continue our investigation in a subsequent paper where we plan to discuss related index and cohomological aspects.

2 Preliminaries on superderivations

Let $\mathcal{H}$ be a (complex) Hilbert space and let Γ be a selfadjoint unitary operator on $\mathcal{H}$. Γ induces a $\mathbb{Z}_2$-grading $\gamma \equiv \mathrm{Ad}\Gamma$ on $B(\mathcal{H})$. We shall denote $B(\mathcal{H})_+$ the unital *-subalgebra of even (Bose) elements of $B(\mathcal{H})$ and by $B(\mathcal{H})_-$ the selfadjoint subspace of odd (Fermi) elements of $B(\mathcal{H})$. Accordingly $B(\mathcal{H}) = B(\mathcal{H})_+ \oplus B(\mathcal{H})_-$. Moreover any γ-invariant subspace $L \subset B(\mathcal{H})$ has a decomposition $L = L_+ \oplus L_-$, where $L_+ \equiv L \cap B(\mathcal{H})_+$ and $L_- \equiv L \cap B(\mathcal{H})_-$.

Now let Q be a selfadjoint operator on $\mathcal{H}$ with domain $D(Q)$ and assume that Q is odd, namely $\Gamma Q \Gamma = -Q$. We now define an operator (*superderivation*) δ on $B(\mathcal{H})$ with domain $D(\delta) \subset B(\mathcal{H})$ as follows.

Let $D(\delta)$ be the set of operators $a \in B(\mathcal{H})$ such that

$$\gamma(a)Q \subset Qa - b, \tag{1}$$

for some bounded operator $b \in B(\mathcal{H})$. Then b is uniquely determined by a and we set $\delta(a) = b$. Clearly $D(\delta)$ is a subspace of $B(\mathcal{H})$ and the map $\delta : D(\delta) \mapsto B(\mathcal{H})$ is linear. Hence we can define a norm $\|\cdot\|_1$ on $D(\delta)$ by

$$\|a\|_1 \equiv \|a\| + \|\delta(a)\|. \tag{2}$$

Note also that $1 \in D(\delta)$ and $\delta(1) = 0$.

We shall now see that $D(\delta)$ is a *-algebra and δ is a superderivation (i.e. a graded derivation). $D(\delta)$ will be called the *domain of the superderivation* $\delta = [Q, \cdot]$. Here the brackets denote the super Lie-algebra brackets induced by Γ on operators on $\mathcal{H}$ (graded commutator).

Proposition 2.1. *The operator δ satisfies the following properties:*

(*i*) *If $a \in D(\delta)$ then $\gamma(a) \in D(\delta)$ and $\delta(\gamma(a)) = -\gamma(\delta(a))$.*

(ii) If $a \in D(\delta)$ then $a^* \in D(\delta)$ and $\delta(a^*) = \gamma(\delta(a)^*)$.

(iii) If $a, b \in D(\delta)$ then $ab \in D(\delta)$ and $\delta(ab) = \delta(a)b + \gamma(a)\delta(b)$.

(iv) δ is a weak–weak closed operator, namely if the net $a_\lambda \in D(\delta)$ converges to $a \in B(\mathcal{H})$ in the weak topology and $\delta(a_\lambda)$ converges to $b \in B(\mathcal{H})$ in the weak topology then $a \in D(\delta)$ and $\delta(a) = b$.

(v) $D(\delta)$ is dense in $B(\mathcal{H})$ in the strong topology.

(vi) If $a, b \in D(\delta)$ then $\|\gamma(a)\|_1 = \|a\|_1$, $\|a^*\|_1 = \|a\|_1$ and $\|ab\|_1 \leq \|a\|_1\|b\|_1$,

Proof. (i) Since $\Gamma Q \Gamma = -Q$ then $\Gamma D(Q) = D(Q)$. Hence, if $a \in D(\delta)$ then $\gamma(a)D(Q) \subset D(Q)$ and a straightforward computation shows that, for every $\psi \in D(Q)$, $Q\gamma(a)\psi - aQ\psi = -\Gamma\delta(a)\Gamma\psi$. Hence, $\gamma(a) \in D(\delta)$ and $\delta(\gamma(a)) = -\gamma(\delta(a))$.
(ii) Let $a \in D(\delta)$ and $\psi_1, \psi_2 \in D(Q)$. Then,

$$\begin{aligned}(a^*\psi_1, Q\psi_2) &= (\psi_1, aQ\psi_2) = -(\psi_1, \delta(\gamma(a))\psi_2) + (\psi_1, Q\gamma(a)\psi_2) \\ &= -(\delta(\gamma(a))^*\psi_1, \psi_2) + (\gamma(a^*)Q\psi_1, \psi_2).\end{aligned}$$

It follows that, $a^*\psi_1 \in D(Q)$ and $Qa^*\psi_1 = \gamma(a^*)Q\psi_1 - \delta(\gamma(a))^*\psi_1$. Hence, since $\psi_1 \in D(Q)$ was arbitrary, $a^* \in D(\delta)$ and $\delta(a^*) = -\delta(\gamma(a))^* = \gamma(\delta(a)^*)$.

(iii) Let $a, b \in D(\delta)$ and $\psi \in D(Q)$. Then $ab\psi, b\psi \in D(Q)$ and

$$\begin{aligned}Qab\psi &= Qab\psi - \gamma(a)Qb\psi + \gamma(a)Qb\psi - \gamma(a)\gamma(b)Q\psi + \gamma(ab)Q\psi \\ &= \delta(a)b\psi + \gamma(a)\delta(b)\psi + \gamma(ab)Q\psi.\end{aligned}$$

Hence $ab \in D(\delta)$ and $\delta(ab) = \delta(a)b + \gamma(a)\delta(b)$.

(iv) Let $a_\lambda \in D(\delta)$ be a net and let $a, b \in B(\mathcal{H})$ be bounded operators such that $\lim a_\lambda = a$ and $\lim \delta(a_\lambda) = b$ in the weak topology of $B(\mathcal{H})$ and let $\psi_1, \psi_2 \in D(Q)$. Then

$$\begin{aligned}(a\psi_1, Q\psi_2) &= \lim(\psi_1, a_\lambda^* Q\psi_2) = -\lim(\psi_1, \delta(\gamma(a_\lambda^*))\psi_2) + \lim(\psi_1, Q\gamma(a_\lambda^*)\psi_2) \\ &= \lim(\psi_1, \delta(a_\lambda)^*\psi_2) + \lim(\gamma(a_\lambda)Q\psi_1, \psi_2) \\ &= (b\psi_1, \psi_2) + (\gamma(a)Q\psi_1, \psi_2).\end{aligned}$$

Hence, $a\psi_1 \in D(Q)$ and $Qa\psi_1 = \gamma(a)Q\psi_1 + b\psi_1$ and since $\psi_1 \in D(Q)$ was arbitrary $a \in D(\delta)$ and $\delta(a) = b$.

(v) Since we have shown that $D(\delta)$ is a unital *-subalgebra of $B(\mathcal{H})$, by von Neumann density theorem it is enough to show that the commutant $D(\delta)'$ contains only the scalar multiples of the identity operator. Let $t \mapsto \alpha_t$ be the (σ-weakly) continuous one-parameter group of automorphisms of $B(\mathcal{H})$ defined by $\alpha_t(a) = e^{itQ}ae^{-itQ}$, $a \in B(\mathcal{H})$ and let $\tilde{\delta}$ be the corresponding generator with domain $D(\tilde{\delta})$, see e.g. [1]. It is well known that $D(\tilde{\delta})$ is a strongly dense unital *-subalgebra of $B(\mathcal{H})$ [1]. Moreover, from the equality $\gamma(\alpha_t(a)) = \alpha_{-t}(\gamma(a))$ it follows that $D(\tilde{\delta})$ is γ-invariant. Thus $D(\tilde{\delta})_+$ is strongly dense in $B(\mathcal{H})_+ = \{\Gamma\}'$ and consequently $(D(\tilde{\delta})_+)' = \{\Gamma\}''$. Now, it follows from [1, Proposition 3.2.55] that $D(\delta)_+ = D(\tilde{\delta})_+$ and that $\delta(a) = -i\tilde{\delta}(a)$ for any $a \in D(\delta)_+$. Hence, $D(\delta)' \subset \{\Gamma\}''$. Now, if $Q = 0$, $D(\delta) = B(\mathcal{H})$ and there is nothing to prove. If $Q \neq 0$ then $Q(Q^2+1)^{-1}$ is a nonzero odd element in $D(\delta)$ and hence $\Gamma \notin D(\delta)'$ so that $D(\delta)' = \mathbb{C}1$.

(vi) The two equalities from the norm follows directly from (i) and (ii). Now let $a, b \in D(\delta)$ then, by (iii) we have $ab \in D(\delta)$ and $\delta(ab) = \delta(a)b + \gamma(a)\delta(b)$. Accordingly

$$\|ab\|_1 \leq \|a\|\|b\| + \|\delta(a)\|\|b\| + \|a\|\|\delta(b)\| \leq \|a\|_1\|b\|_1.$$

$\square$

Remark 2.2. It follows from (iv) of the above proposition that δ is $\mathcal{T}-\mathcal{T}$ closed if $\mathcal{T}$ is the strong, σ-weak or σ-strong topology. Indeed, as any of such a topology $\mathcal{T}$ is stronger than the weak topology, the graph of δ is closed in the $\mathcal{T}$-topology of $B(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$ too.

Corollary 2.3. *$D(\delta)$ with the norm $\|\cdot\|_1$ is a unital Banach *-algebra.*

The following lemma will be useful later.

Lemma 2.4. *Let $D \subset D(Q)$ be a core for Q let $a \in B(\mathcal{H})$ and assume that $aD \subset D(Q)$ and the map $D \ni \psi \mapsto Qa\psi - \gamma(a)Q\psi$ extends to a bounded linear operator $b \in B(\mathcal{H})$. Then $a \in D(\delta)$ and $\delta(a) = b$.*

Proof. Let $\psi \in D(Q)$. By assumption D is a core for Q and thus there is a sequence $\psi_n \in D$ such that $\lim \psi_n = \psi$ and $\lim Q\psi_n = Q\psi$. Hence, $\lim a\psi_n = a\psi$ and $\lim Qa\psi_n = \gamma(a)Q\psi + b\psi$ and since Q, being selfadjoint, is a closed operator, $a\psi \in D(Q)$ and $Qa\psi = \gamma(a)Q\psi + b\psi$. Since $\psi \in D(Q)$ was arbitrary we have proved that $\gamma(a)Q \subset Qa - b$ and the conclusion follows. $\square$

We now consider the domains $D(\delta^n)$, $n \in \mathbb{N}$, of the powers of δ. Note that $D(\delta) = D(\delta^1) \supset D(\delta^2) \supset D(\delta^3) \ldots$ and that $C^\infty(\delta) \equiv \cap_{n=1}^\infty D(\delta^n)$ is δ-invariant.

We define a norm $\|\cdot\|_n$ on $D(\delta^n)$, $n \in \mathbb{N}$ by (2) and the recursive relation

$$\|a\|_{n+1} = \|a\|_n + \|\delta(a)\|_n. \tag{3}$$

Proposition 2.5. *The subspaces $D(\delta^n)$, $n \in \mathbb{N}$ and $C^\infty(\delta)$ are γ-invariant unital *-subalgebras of $B(\mathcal{H})$. Moreover the pair $\big(D(\delta^n), \|\cdot\|_n\big)$ is a Banach *-algebra for all $n \in \mathbb{N}$ and $\|\gamma(a)\|_n = \|a\|_n$ for all $a \in D(\delta^n)$.*

Proof. Clearly it is enough to prove the proposition for the subspaces $D(\delta^n)$, $n \in \mathbb{N}$. We proceed by induction.

From Proposition 2.1 $D(\delta^1) = D(\delta)$, is a γ-invariant unital *-subalgebra of $B(\mathcal{H})$. Moreover, it is a Banach *-algebra with the norm $\|\cdot\|_1$. Assume now that the same is true for the pair $\big(D(\delta^n), \|\cdot\|_n\big)$. Since $1 \in D(\delta)$ and $\delta(1) = 0$ we can conclude that $1 \in D(\delta^{n+1})$. Now let $a, b \in D(\delta^{n+1})$. Then, $\gamma(a), a^*, ab \in D(\delta)$ and $\delta(a), \delta(b) \in D(\delta^n)$. Moreover $\delta(\gamma(a)) = -\gamma(\delta(a)) \in D(\delta^n)$, $\delta(a^*) = \gamma(\delta(a)^*) \in D(\delta^n)$ and $\delta(ab) = \delta(a)b + \gamma(a)\delta(b) \in D(\delta^n)$. Hence, $D(\delta^{n+1})$ is a unital γ-invariant *-subalgebra of $B(\mathcal{H})$. That the norm $\|\cdot\|_{n+1}$ is a *-algebra norm $D(\delta^{n+1})$ and that it is γ-invariant follows exactly as in the proof of Proposition 2.1 (vi) and it remains to show that $D(\delta^{n+1})$ is complete. Let a_m, $m \in \mathbb{N}$ be a Cauchy sequence in $D(\delta^{n+1})$. By the inductive assumption a_m and $\delta(a_m)$ converge to elements $a, b \in D(\delta^n)$ respectively and it follows from Proposition 2.1 (iv) that $b = \delta(a)$ and hence $a \in D(\delta^{n+1})$ and $\|a_m - a\|_{n+1}$ tends to 0 as m tends to ∞. $\square$

For every $a \in B(\mathcal{H})$ we denote by $\sigma(a)$ the spectrum of a. The following proposition can be proved adapting the proof of [1, Proposition 3.2.29].

Proposition 2.6. *If $a \in D(\delta)$ and $\lambda \notin \sigma(a)$ then $(a-\lambda 1)^{-1} \in D(\delta)$ and*

$$\delta\big((a-\lambda 1)^{-1}\big) = -(\gamma(a)-\lambda 1)^{-1}\delta(a)(a-\lambda 1)^{-1}. \tag{4}$$

Corollary 2.7. *For all $n \in \mathbb{N}$, if $a \in D(\delta^n)$ and $\lambda \notin \sigma(a)$ then $(a-\lambda 1)^{-1} \in D(\delta^n)$.*

Corollary 2.8. *For all $n \in \mathbb{N}$, if $a \in D(\delta^n)$ and f is a complex function holomorphic in a neighbourhood of $\sigma(a)$ then $f(a) \in D(\delta^n)$.*

Now consider the positive selfadjoint operator $H \equiv Q^2$ and the corresponding derivation δ_0 on $B(\mathcal{H})$. Then the generator of the one-parameter group of automorphisms $\mathrm{Ad}e^{itH}$ of $B(\mathcal{H})$ is $i\delta_0$. Note that H commutes with Γ.

Lemma 2.9. *If $a \in D(\delta^2)$ then $a \in D(\delta_0)$ and $\delta^2(a) = \delta_0(a)$.*

Proof. Assume that $a \in D(\delta^2)$ and that $\psi \in D(H) \subset D(Q)$. Then $a\psi \in D(Q)$ and $Qa\psi - \gamma(a)Q\psi = \delta(a)\psi$. Now, $Q\psi \in D(Q)$ and moreover, since $\gamma(a) \in D(\delta)$ and $\delta(a) \in D(\delta)$ we have $\delta(a)\psi + \gamma(a)Q\psi \in D(Q)$. Hence $a\psi \in D(H)$ and

$$\begin{aligned} Ha\psi &= Q^2 a\psi = Q\delta(a)\psi + Q\gamma(a)Q\psi \\ &= \delta^2(a)\psi + \gamma(\delta(a))Q\psi + \delta(\gamma(a))Q\psi + aH\psi \\ &= \delta^2(a)\psi + aH\psi. \end{aligned}$$

Since $\psi \in D(H)$ was arbitrary, the conclusion follows from [1, Proposition 3.2.55]. □

For any $f \in L^1(\mathbb{R})$ on $\mathbb{R}$ and any $a \in B(\mathcal{H})$ we define

$$a_f \equiv \int_{\mathbb{R}} e^{itH} a e^{-itH} f(t)\mathrm{d}t. \tag{5}$$

Lemma 2.10. *Assume that $a \in D(\delta)$ and that $f \in L^1(\mathbb{R})$. Then $a_f \in D(\delta)$ and $\delta(a_f) = \delta(a)_f$.*

Proof. This is a straightforward consequence of the fact that the one-parameter group of unitaries e^{itH} commutes with Q and Γ. □

Lemma 2.11. *Assume that $a \in D(\delta)$ and that $f \in C_c^\infty(\mathbb{R})$. Then $a_f \in D(\delta^2)$ and $\delta^2(a_f) = ia_{f'}$.*

Proof. For any $\psi \in D(H)$, we have $Q\psi \in D(Q)$. Moreover, a standard and straightforward argument shows that the map $t \mapsto e^{iHt}a_f e^{-iHt} \in B(\mathcal{H})$ is differentiable at $t = 0$ and that the corresponding derivative is equal to $-a_{f'}$. Hence, by [1, Proposition 3.2.55], $a_f\psi \in D(H)$ and

$Ha_f\psi = ia_{f'}\psi + a_f H\psi$. Similarly $\delta(a)_f\psi \in D(H)$. It follows that $\delta(a_f)\psi = \delta(a)_f\psi \in D(Q)$ and

$$\begin{aligned} Q\delta(a_f)\psi &= Ha_f\psi - Q\gamma(a)_f Q\psi \\ &= Ha_f\psi - \delta(\gamma(a_f))Q\psi - a_f H\psi \\ &= \delta_0(a_f)\psi - \delta(\gamma(a_f))Q\psi \\ &= ia_{f'}\psi - \delta(\gamma(a_f))Q\psi \\ &= ia_{f'}\psi + \gamma(\delta(a_f))Q\psi. \end{aligned}$$

Since $D(H)$ is a core for Q it follows by Lemma 2.4 that $a_f \in D(\delta^2)$ and $\delta^2(a_f) = ia_{f'}$. $\square$

From Lemma 2.10 and Lemma 2.11 the following proposition can be easily proved by induction.

Proposition 2.12. *Assume that $a \in D(\delta)$ and that $f \in C_c^\infty(\mathbb{R})$. Then $a_f \in C^\infty(\delta)$.*

Corollary 2.13. *$C^\infty(\delta)$ is a core for δ (with respect to the σ-weak topology), namely δ coincides with the (σ-weak)–(σ-weak) closure of its restriction to $C^\infty(\delta)$.*

Corollary 2.14. *$C^\infty(\delta)$ is dense in $B(\mathcal{H})$ in the strong topology.*

3 Spectral triples in conformal field theory

The purpose of this section is to state our definitions of spectral triple and give a few comments on Connes definition and related matters.

Firstly we will state the definitions suitable for the Ramond algebra case.

Definition 3.1. A (θ-summable) *graded spectral triple* $(\mathfrak{A}, \mathcal{H}, Q)$ consists of a graded Hilbert space $\mathcal{H}$, where the selfadjoint grading unitary is denoted by Γ, a unital *-algebra $\mathfrak{A} \subset B(\mathcal{H})$ graded by $\gamma \equiv \mathrm{Ad}(\Gamma)$, and an odd selfadjoint operator Q on $\mathcal{H}$ as follows:

- $\mathfrak{A}$ is contained in $D(\delta)$, the domain of the superderivation $\delta = [Q, \cdot\,]$ as in Sect. 2;
- For every $\beta > 0$, $\mathrm{Tr}(e^{-\beta Q^2}) < \infty$ (θ-summability).

The operator Q is called the *supercharge* operator, its square the *Hamiltonian*.

Remark 3.2. Restricting to the even subalgebra $\mathfrak{A}_+$ of $\mathfrak{A}$, the above definition is essentially Connes [6] (see also [7]) definition of a (even) spectral triple $(\mathfrak{A}_+, \mathcal{H}, Q)$. This is the fundamental object for index theorems and evaluating on K-theory elements. In this case the supercharge Q is traditionally called *Dirac operator* and denoted by D.

Remark 3.3. Let $\mathcal{H}$ Hilbert space graded by Γ and let $\mathcal{A}$ be a unital *-subalgebra of $B(\mathcal{H})$ such that $\gamma(\mathcal{A}) = \mathcal{A}$. Let moreover $\mathfrak{A} \equiv \mathcal{A} \cap C^\infty(\delta)$. Then, provided $\mathrm{Tr}(e^{-\beta Q^2}) < \infty$ for all $\beta > 0$, $(\mathfrak{A}, \mathcal{H}, Q)$ is a graded spectral triple in the sense of Definition 3.1. Note also that if $\mathcal{A}$ is a von Neumann algebra then δ restricts to a weak-weak closed superderivation of $\mathcal{A}$.

Our spectral triples will satisfy an additional property which is described in the following definition taken from [13].

Definition 3.4. A *quantum algebra* $(\mathfrak{A}, \mathcal{H}, Q)$ is a (θ-summable) graded spectral triple such that $\delta(\mathfrak{A}) \subset \mathfrak{A}$.

Remark 3.5. Let $(\mathfrak{A}, \mathcal{H}, Q)$ be a quantum algebra, thus the additional property $\delta(\mathfrak{A}) \subset \mathfrak{A}$ is satisfied, and let $J : \mathcal{H} \mapsto \mathcal{H}$ be an antiunitary involution such that $J\mathfrak{A}J \subset \mathfrak{A}'$. Then we have $J\mathfrak{A}J \subset \delta(\mathfrak{A})'$ which is essentially the order one condition for the operator Q (see e.g. [7]). If we restrict to the associated even spectral triples we have $\delta(\mathfrak{A}_+) \subset \mathfrak{A}_-$ and hence $\delta(\mathfrak{A}_+) \cap \mathfrak{A}_+ = \{0\}$. However $J\mathfrak{A}_+J \subset \delta(\mathfrak{A})' \subset \delta(\mathfrak{A}_+)'$ and the order one condition for Q is still satisfied.

Remark 3.6. If $(\mathfrak{A}, \mathcal{H}, Q)$ is a quantum algebra clearly we have $\mathfrak{A} \subset C^\infty(\delta)$. Conversely let $\mathcal{H}$ and $\mathcal{A}$ as in Remark 3.3. Suppose that $\delta(a) \in \mathcal{A}$ for every $a \in \mathcal{A} \cap D(\delta)$ and let $\mathfrak{A} \equiv \mathcal{A} \cap C^\infty(\delta)$. Then, provided $\mathrm{Tr}(e^{-\beta Q^2}) < \infty$ for all $\beta > 0$, $(\mathfrak{A}, \mathcal{H}, Q)$ is a quantum algebra.

Remark 3.7. The supercharge operator Q appears in supersymmetric field theories and its square Q^2 is the Hamiltonian. In conformal field theory, the subject of this paper, it will be (up to an additive constant) the conformal Hamiltonian L_0^λ in the considered representation λ, c.f. Section 4. Then the θ-summability condition is automatically satisfied under very general conditions.

In the Neveu-Schwarz case, see Section 5, we will have a Hamiltonian H and a superderivation δ on the algebra $\mathfrak{A}$, *without* a supercharge operator. Namely there can be no odd selfadjoint operator Q satisfying $Q^2 = H$ and $\delta = [Q, \cdot]$. To treat also this case we need to generalise the definition of spectral triple. However to express the condition $Q^2 = H$ in terms of the superderivation δ we need to give a meaning to its square δ^2. We are thus led to assume from the beginning the additional condition $\delta(\mathfrak{A}) \subset \mathfrak{A}$ and thus to generalise only the notion of quantum algebra. This will suffice for the purposes of this paper.

Definition 3.8. A *generalised quantum algebra* $(\mathfrak{A}, \mathcal{H}, \delta)$ consists of a graded Hilbert space $\mathcal{H}$, where the selfadjoint grading unitary is denoted by Γ, a unital *-algebra $\mathfrak{A} \subset B(\mathcal{H})$ graded by $\gamma \equiv \mathrm{Ad}(\Gamma)$, and an antisymmetric odd superderivation $\delta : \mathfrak{A} \to \mathfrak{A}$, i.e., a linear map satisfying

$$\begin{aligned} \delta(a^*) &= -\delta(\gamma(a))^* \\ \delta(\gamma(a)) &= -\gamma(\delta(a)) \\ \delta(ab) &= \delta(a)b + \gamma(a)\delta(b) \end{aligned}$$

$a, b \in D(\delta)$, with the following properties:

- δ is σ-weakly closable, i.e. it extends to a (σ-weakly)–(σ-weakly) closed superderivation of the von Neumann algebra $\mathfrak{A}''$.
- There exists an even positive selfadjoint operator H on $\mathcal{H}$ (the Hamiltonian) such that for every $a \in \mathfrak{A}$ and every $\psi \in D(H)$, $a\psi \in D(H)$ and $Ha\psi - aH\psi = \delta^2(a)\psi$.
- For every $\beta > 0$, the operator $e^{-\beta H}$ is of trace class.

In the following two sections we will construct spectral triples of the above types. Indeed we shall have nets of spectral triples in the following sense. Let $\mathcal{I}$ be the family of nonempty, nondense, open intervals of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let

$$\mathcal{I}_0 \equiv \{I \in \mathcal{I} : \overline{I} \subset S^1 \setminus \{-1\}\},$$

where $\overline{I}$ denotes the closure of the interval $I \in \mathcal{I}$.

Definition 3.9. A *net of graded spectral triples* $(\mathfrak{A}, \mathcal{H}, Q)$ on S^1 (resp. $S^1 \setminus \{-1\}$) consists of graded Hilbert space $\mathcal{H}$, an odd selfadjoint operator Q and a net $\mathfrak{A}$ of unital *-algebras on $\mathcal{I}$ (resp. $\mathcal{I}_0$) acting on $\mathcal{H}$, i.e. a map from $\mathcal{I}$ (resp. $\mathcal{I}_0$) into the family of unital *-subalgebras of $B(\mathcal{H})$ which satisfies isotony property

$$\mathfrak{A}(I_1) \subset \mathfrak{A}(I_2) \quad \text{if } I_1 \subset I_2,$$

such that $(\mathfrak{A}(I), \mathcal{H}, Q)$ is a graded spectral triple for all $I \in \mathcal{I}$ (resp. $I \in \mathcal{I}_0$). If the net satisfies the additional property $\delta(\mathfrak{A}(I)) \subset \mathfrak{A}(I)$, $\delta = [Q, \cdot]$, for all $I \in \mathcal{I}$ (resp. $I \in \mathcal{I}_0$) then we say that $(\mathfrak{A}, \mathcal{H}, Q)$ is a *net of quantum algebras* on S^1 (resp. $S^1 \setminus \{-1\}$).

Now we give a more general definition to cover the case where there is no global supercharge operator. In this context the nets will be on the double cover of S^1, [4, Sect. 3.2]. Denote by $\mathcal{I}^{(n)}$ the intervals on the n-cover $S^{1(n)}$ of S^1, namely $I \in \mathcal{I}^{(n)}$ if I is a connected subset of $S^{1(n)}$ whose projection onto the base S^1 belongs to $\mathcal{I}$.

Definition 3.10. A *net of generalised quantum algebras* $(\mathfrak{A}, \mathcal{H}, \delta)$ on $S^{1(n)}$ (resp. $S^1 \setminus \{-1\}$) consists of a graded Hilbert space $\mathcal{H}$, of a net of unital *-algebras on $S^{1(n)}$ (resp. $S^1 \setminus \{-1\}$) acting on $\mathcal{H}$ and a net δ of superderivations on $\mathfrak{A}$, i.e. a map $I \in \mathcal{I} \mapsto \delta_I$ (resp. $I \in \mathcal{I}_0 \mapsto \delta_I$), where $\delta_I : \mathfrak{A}(I) \mapsto \mathfrak{A}(I)$ is a superderivation, satisfying $\delta_{I_2}|_{\mathfrak{A}(I_1)} = \delta_{I_1}$ if $I_1 \subset I_2$, such that $(\mathfrak{A}(I), \mathcal{H}, \delta_I)$ is a generalised quantum algebra for every $I \in \mathcal{I}$ (resp. $I \in \mathcal{I}_0$) with Hamiltonian H independent of I.

Note that a net on $S^{1(n)}$ gives rise to a net on $S^1 \setminus \{-1\}$ by restriction. Conversely, if rotation covariance holds true, a net on $S^1 \setminus \{-1\}$ extends to a net on $S^{1(n)}$ for some finite or infinite n.

4 Spectral triples from the Ramond algebra

In this section we shall construct nets of quantum algebras from representations of the Ramond (Super-Virasoro) algebra.

Recall that the Ramond algebra is the super-Lie algebras generated by even elements L_n, $n \in \mathbb{Z}$, odd elements G_r, $r \in \mathbb{Z}$, and a central even element k satisfying the relations

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{k}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, G_r] &= (\frac{m}{2} - r)G_{m+r}, \\ [G_r, G_s] &= 2L_{r+s} + \frac{k}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}. \end{aligned} \tag{6}$$

We shall consider representations λ of the Ramond algebra by linear endomorphisms, denoted by $L^\lambda_m, G^\lambda_r, k^\lambda$, $m, r \in \mathbb{Z}$, of a complex vector space V_λ equipped with an involutive linear endomorphism Γ_λ inducing the super-Lie algebra grading. The endomorphisms $L^\lambda_m, G^\lambda_r, k^\lambda$, satisfies the relations 6 with respect to the brackets given by the supercommutator induced by Γ_λ.

We assume that the representation λ satisfies the following properties.

(i) $k^\lambda = c1$ for some $c \in \mathbb{C}$ (the *central charge* of the representation λ).
(ii) L_0^λ is diagonalizable on V_λ, namely

$$V_\lambda = \bigoplus_{\alpha \in \mathbb{C}} \mathrm{Ker}(L_0^\lambda - \alpha 1) \tag{7}$$

(iii) $\mathrm{Ker}(L_0^\lambda - \alpha 1)$ is finite-dimensional for all $\alpha \in \mathbb{C}$.

(iv) (*Unitarity*) There is a scalar product $(\cdot, \cdot)$ on V_λ such that

$$\begin{aligned}(L_m^\lambda u, v) &= (u, L_{-m}^\lambda v),\\ (G_r^\lambda u, v) &= (u, G_{-r}^\lambda v),\\ (\Gamma_\lambda u, v) &= (u, \Gamma_\lambda v),\end{aligned}$$

for all $u, v \in V_\lambda$ and all $m, r \in \mathbb{Z}$.

As a consequence of the above assumptions the central charge c is a real number and

$$\mathrm{Ker}(L_0^\lambda - \alpha 1) = \{0\}$$

if α is not a real number such that $\alpha \geq c/24$. Since the eigenvalues of L_0^λ are real numbers bounded from below it follows from the (unitary) representation theory of the Virasoro algebra that $c \geq 0$ and hence λ is a *positive energy representation*. Accordingly, the possible values of the central charge are either $c \geq 3/2$ or

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right),\ m = 2, 3, \ldots \tag{8}$$

see [9].

Note that the graded unitary lowest weight representations of the Ramond algebra satisfy all the above assumptions.

Now let λ be a representation of the Ramond algebra satisfying assumptions (i)–(iv) and let $\mathcal{H}_\lambda$ be the Hilbert space completion of V_λ. The endomorphisms L_m^λ, G_r^λ, $m, r \in \mathbb{Z}$, define unbounded operators on $\mathcal{H}_\lambda$ with domain V_λ, which are closable since by assumption (iv) they have densely defined adjoint. We shall denote their closure by the same symbols. With this convention L_0^λ is a selfadjoint operator on $\mathcal{H}_\lambda$. Moreover Γ_λ extends to a selfadjoint unitary operator on $\mathcal{H}_\lambda$ which will also be denoted by the same symbol.

The operators L_m^λ, G_r^λ, $m, r \in \mathbb{Z}$ satisfy the *energy bounds*

$$\|L_m^\lambda v\| \leq M(1 + |m|^{\frac{3}{2}})\|(1 + L_0^\lambda)v\|, \quad v \in V_\lambda, \tag{9}$$

for a suitable constant $M > 0$ depending on the central charge c and

$$\|G_r^\lambda v\| \leq (2 + \frac{c}{3}r^2)^{\frac{1}{2}}\|(1 + L_0^\lambda)^{\frac{1}{2}}v\|, \quad v \in V_\lambda, \tag{10}$$

see [4, Sect. 6.3] and the references therein.

Now let f be a smooth function on S^1. It follows from the linear energy bounds in Equations (9), (10) and the fact that the Fourier coefficients

$$\hat{f}_n = \int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}\frac{\mathrm{d}\theta}{2\pi},\ n \in \mathbb{Z}, \tag{11}$$

are rapidly decreasing, that the maps

$$V_\lambda \ni v \mapsto \sum_{n\in\mathbb{Z}} \hat{f}_n L_n^\lambda v \tag{12}$$

$$V_\lambda \ni v \mapsto \sum_{r\in\mathbb{Z}} \hat{f}_r G_r^\lambda v \tag{13}$$

define closable operators on $\mathcal{H}_\lambda$ and we shall denote by $L^\lambda(f)$ and $G^\lambda(f)$ (*smeared fields*) respectively the corresponding closures. Their domains contain $D(L_0^\lambda)$ and they leave invariant $C^\infty(L_0^\lambda)$. Moreover, if f is real, $L^\lambda(f)$ and $G^\lambda(f)$ are selfadjoint operators which are essentially selfadjoint on any core for L_0^λ, cf. [3].

Using these smeared fields we shall define a net of von Neumann algebras in the usual way. Let $\mathcal{I}$ as in Section 3. We define a net $\mathcal{A}_\lambda$ of von Neumann algebras on S^1 by

$$\mathcal{A}_\lambda(I) \equiv \{e^{iL^\lambda(f)}, e^{iG^\lambda(f)} : f \in C^\infty(S^1) \text{ real, supp} f \subset I\}'', \ I \in \mathcal{I}. \tag{14}$$

It is clear from the definition that isotony is satisfied, namely

$$\mathcal{A}_\lambda(I_1) \subset \mathcal{A}_\lambda(I_2) \quad \text{if } I_1 \subset I_2. \tag{15}$$

In the same way, we see that $L^\lambda(f)$ and $G^\lambda(f)$ are affiliated with $\mathcal{A}_\lambda(I)$ if supp$f \subset I$. Moreover each algebra $\mathcal{A}_\lambda(I)$, $I \in \mathcal{I}$, is left globally invariant by the grading automorphism $\gamma_\lambda \equiv \mathrm{Ad}\Gamma_\lambda$ of $B(\mathcal{H}_\lambda)$.

Let $\mathrm{Diff}(S^1)$ be the group of (smooth) orientation preserving diffeomorphisms of S^1 and let $\mathrm{Diff}^{(\infty)}(S^1)$ be its universal cover. Moreover let $\mathrm{Diff}_I(S^1) \subset \mathrm{Diff}(S^1)$ be the subgroup of diffeomorphisms that are localised in I, namely that act trivially on I', and let $\mathrm{Diff}_I^{(\infty)}(S^1)$ be the connected component of the identity of the pre-image of $\mathrm{Diff}_I(S^1)$ in $\mathrm{Diff}^{(\infty)}(S^1)$. We denote by $u_\lambda : \mathrm{Diff}^\infty(S^1) \mapsto U(\mathcal{H}_\lambda)/\mathbb{T}$ the strongly continuous projective unitary positive energy representation obtained by integrating the restriction of the representation λ to the Virasoro Lie subalgebra of the Ramond algebra, see [11, 19] and for every $g \in \mathrm{Diff}^\infty(S^1)$ we choose a unitary operator $U_\lambda(g)$ in the equivalence class $u_\lambda(g) \in U(\mathcal{H}_\lambda)/\mathbb{T}$. If $g \in \mathrm{Diff}_I^{(\infty)}(S^1)$ one can show that $U_\lambda(g) \in \mathcal{A}_\lambda(I)$, see the proof of [4, Theorem 33]. Note that if λ is a lowest weight representation with lowest weight h_λ then $e^{i2\pi L_0^\lambda} = e^{i2\pi h_\lambda}$ and the projective unitary representation u_λ factors through $\mathrm{Diff}(S^1)$.

Arguing as in [4, Sect. 6.3] where a similar construction has been carried out for the vacuum representations of the Neveu-Schwarz (super-Virasoro) algebra we obtain the following theorem. We shall omit the details of the proof which in the case of the Ramond algebra are similar but in fact simpler.

Theorem 4.1. *The net $\mathcal{A}_\lambda$ satisfies the following properties:*

- (i) *(Graded locality) If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \varnothing$ then $\mathcal{A}_\lambda(I_1) \subset Z_\lambda \mathcal{A}_\lambda(I_2)' Z_\lambda^*$, where $Z_\lambda \equiv (1 - i\Gamma_\lambda)/(1-i)$.*
- (ii) *(Conformal covariance) $U_\lambda(g)\mathcal{A}_\lambda(I)U_\lambda(g)^* = \mathcal{A}_\lambda(\dot{g}I)$, for all $g \in \mathrm{Diff}^{(\infty)}(S^1)$ and all $I \in \mathcal{I}$, where $\dot{g}$ denotes the image of g in $\mathrm{Diff}(S^1)$ under the covering map.*

Note that for any λ (not necessarily a lowest weight representation) we have

$$e^{i2\pi L_0^\lambda} \in \bigcap_{I\in\mathcal{I}} \mathcal{A}_\lambda(I)' \tag{16}$$

and hence the action of $\mathrm{Diff}^{(\infty)}(S^1)$ on the local algebras factors through $\mathrm{Diff}(S^1)$.

We now define a supercharge operator Q_λ on $\mathcal{H}_\lambda$ by $Q_\lambda \equiv G_0^\lambda$. Q_λ is an odd operator, namely $\Gamma_\lambda Q_\lambda \Gamma_\lambda = -Q_\lambda$. Moreover it satisfies

$$Q_\lambda^2 = H_\lambda \equiv L_0^\lambda - \frac{c}{24}. \tag{17}$$

We denote by δ the corresponding superderivation as defined in Section 2. We now define a net of unital *-subalgebras on S^1 by $\mathfrak{A}_\lambda(I) \equiv \mathcal{A}_\lambda(I) \cap C^\infty(\delta)$, $I \in \mathcal{I}$. Our aim is to show that this net satisfies $\delta(\mathfrak{A}_\lambda(I)) \subset \mathfrak{A}_\lambda(I)$ and that it is strong operator dense in $\mathcal{A}_\lambda(I)$ for any $I \in \mathcal{I}$. This will give a net of quantum algebras on S^1 naturally associated with the net of von Neumann algebras $\mathcal{A}_\lambda$ and hence a noncommutative geometric structure [6] associated to superconformal quantum field theories. Inspired by the work of Buchholz and Grundling [2] we shall use resolvents of the smeared Bose fields to exhibit local elements in the domain of δ. Yet the models considered here appear to be more complicated than the free field model considered by them.

According with the notation in Sect. 2 we set, for all $a \in B(\mathcal{H}_\lambda)$ and $f \in C_c^\infty(\mathbb{R})$,

$$a_f \equiv \int_{\mathbb{R}} e^{itH_\lambda} a e^{-itH_\lambda} f(t)\mathrm{d}t = \int_{\mathbb{R}} e^{itL_0^\lambda} a e^{-itL_0^\lambda} f(t)\mathrm{d}t. \tag{18}$$

Proposition 4.2. *$\delta(\mathfrak{A}_\lambda(I)) \subset \mathfrak{A}_\lambda(I)$ for all $I \in \mathcal{I}$.*

Proof. Fix an arbitrary interval $I \in \mathcal{I}$ and an arbitrary $a \in \mathcal{A}_\lambda(I) \cap D(\delta)$. As a consequence of Remark 3.6 it is enough to show that $\delta(a) \in \mathcal{A}_\lambda(I)$. Let $I_1, I_2 \in \mathcal{I}$ be such that the closure $\overline{I}$ of I is contained in I_1 and $\overline{I_1} \subset I_2$. Then, if the support of $f \in C_c^\infty(\mathbb{R})$ is sufficiently close to 0, $a_f \in \mathcal{A}_\lambda(I_1)$. By Lemma 2.10, $a_f \in D(\delta)$ and $\delta(a_f) = \delta(a)_f$. Moreover, a standard argument shows that $a_f C^\infty(L_0^\lambda) \subset C^\infty(L_0^\lambda)$.

Now let φ_1 and φ_2 be two real nonnegative smooth functions on S^1 such that $\mathrm{supp}\varphi_1 \subset I_2$, $\mathrm{supp}\varphi_2 \subset I_1'$ and $\varphi_1 + \varphi_2 = 1$ and let $\psi \in C^\infty(L_0^\lambda)$. Then,

$$\begin{aligned} \delta(a)_f\psi &= \delta(a_f)\psi = Q_\lambda a_f \psi - \gamma_\lambda(a_f)Q_\lambda\psi \\ &= G^\lambda(\varphi_1)a_f\psi + G^\lambda(\varphi_2)a_f\psi - \gamma_\lambda(a_f)G^\lambda(\varphi_1)\psi - \gamma_\lambda(a_f)G^\lambda(\varphi_2)\psi. \end{aligned}$$

Since $G^\lambda(\varphi_2)$ is affiliated with $\mathcal{A}_\lambda(I_1') \subset Z_\lambda \mathcal{A}_\lambda(I_1)' Z_\lambda^*$ (using graded locality), we have

$$G^\lambda(\varphi_2)a_f\psi - \gamma_\lambda(a_f)G^\lambda(\varphi_2)\psi = 0.$$

Hence,

$$\delta(a)_f\psi = G^\lambda(\varphi_1)a_f\psi - \gamma_\lambda(a_f)G^\lambda(\varphi_1)\psi.$$

Then, given an arbitrary $b \in \mathcal{A}_\lambda(I_2)'$, we have

$$\begin{aligned} b\delta(a)_f\psi &= bG^\lambda(\varphi_1)a_f\psi - b\gamma_\lambda(a_f)G^\lambda(\varphi_1)\psi \\ &= G^\lambda(\varphi_1)a_f b\psi - \gamma_\lambda(a_f)G^\lambda(\varphi_1)b\psi. \end{aligned}$$

Since $C^\infty(L_0^\lambda)$ is a core for $G^\lambda(\varphi_1)$ we can find a sequence $\psi_n \in C^\infty(L_0^\lambda)$ such that ψ_n tends to $b\psi$ and $G^\lambda(\varphi_1)\psi_n$ tends to $G^\lambda(\varphi_1)b\psi$ as n tends to ∞. Then

$$\begin{aligned}\lim_{n\to\infty} G^\lambda(\varphi_1)a_f\psi_n &= \lim_{n\to\infty}\Big(\delta(a)_f\psi_n + \gamma_\lambda(a_f)G^\lambda(\varphi_1)\psi_n\Big)\\ &= \delta(a)_f b\psi + \gamma_\lambda(a_f)G^\lambda(\varphi_1)b\psi\end{aligned}$$

and since $G^\lambda(\varphi_1)$ is a closed operator it follows that

$$G^\lambda(\varphi_1)a_f b\psi = \delta(a)_f b\psi + \gamma_\lambda(a_f)G^\lambda(\varphi_1)b\psi.$$

Hence

$$\begin{aligned}b\delta(a)_f\psi &= G^\lambda(\varphi_1)a_f b\psi - \gamma_\lambda(a_f)G^\lambda(\varphi_1)b\psi\\ &= \delta(a)_f b\psi + \gamma_\lambda(a_f)G^\lambda(\varphi_1)b\psi - \lim_{n\to\infty}\gamma_\lambda(a_f)G^\lambda(\varphi_1)\psi_n\\ &= \delta(a)_f b\psi.\end{aligned}$$

It follows that $\delta(a)_f \in \mathcal{A}_\lambda(I_2)$ for every smooth function f on $\mathbb{R}$ with support sufficiently close to 0. Hence, $\delta(a) \in \mathcal{A}_\lambda(I_2)$ and since I_2 can be any interval in $\mathcal{I}$ containing the closure of I,

$$\delta(a) \in \bigcap_{I_0 \supset \overline{I}} \mathcal{A}_\lambda(I_0).$$

The conclusion follows since the latter intersection of von Neumann algebras coincides with $\mathcal{A}_\lambda(I)$ as a consequence of conformal covariance. $\square$

To show that $\mathfrak{A}_\lambda(I)$ is strong operator dense in $\mathcal{A}_\lambda(I)$ for all $I \in \mathcal{I}$ we need some preliminary results.

Proposition 4.3. *For every $k \in \mathbb{N}$ and every real $f \in C^\infty(S^1)$ there exists a real number $M > 0$ such that, for every $\alpha \in \mathbb{R}$ satisfying $|\alpha| > M$ the following holds*

$$(L^\lambda(f) + i\alpha)^{-1}D((L_0^\lambda)^k) \subset D((L_0^\lambda)^k).$$

Proof. Let $(\cdot,\cdot)_k$ be the scalar product on $D((L_0^\lambda)^k)$ given by

$$(\psi_1,\psi_2)_k \equiv ((L_0^\lambda+1)^k\psi_1, (L_0^\lambda+1)^k\psi_2).$$

With this scalar product $D((L_0^\lambda)^k)$ is a Hilbert space which we shall denote by $\mathcal{H}^k$. Let $\|\cdot\|_k$ be the corresponding norm. By [19, Proposition 2.1], $e^{itL^\lambda(f)}$, $t \in \mathbb{R}$, restricts to bounded linear maps $\mathcal{H}^k \to \mathcal{H}^k$ satisfying

$$\|e^{itL^\lambda(f)}\|_{B(\mathcal{H}^k)} \le e^{|t|M},$$

for suitable constant $M > 0$ (depending on f and k). Moreover, it follows from [19, Corollary 2.3] (see also [19, Lemma 3.1.1]) that the map $t \to e^{itL^\lambda(f)} \in B(\mathcal{H}^k)$ is strongly continuous.

Now let $\alpha > M$ and let $\psi \in D((L_0^\lambda)^k)$ then on $\mathcal{H}_\lambda$ we have the equality

$$(L^\lambda(f) + i\alpha)^{-1}\psi = -i\int_0^\infty e^{itL^\lambda(f)}e^{-t\alpha}\psi \mathrm{d}t.$$

The map $t \to e^{itL^\lambda(f)}e^{-t\alpha}\psi \in \mathcal{H}^k$ is continuous and

$$\int_0^\infty \|e^{itL^\lambda(f)}e^{-t\alpha}\psi\|_k \mathrm{d}t \leq \|\psi\|_k \int_0^\infty e^{(M-\alpha)t}\mathrm{d}t < \infty.$$

Hence

$$(L^\lambda(f) + i\alpha)^{-1}\psi \in \mathcal{H}^k = D((L_0^\lambda)^k).$$

A similar argument shows that $(L^\lambda(f)+i\alpha)^{-1}\psi \in D((L_0^\lambda)^k)$ also if $\alpha < -M$ completing the proof. $\square$

In the following, for every differentiable function f on S^1, we shall denote by f' the function on S^1 defined by $f'(e^{i\theta}) = \frac{\mathrm{d}}{\mathrm{d}\theta}f(e^{i\theta})$. Moreover if f is any integrable function on S^1 we shall use the notation $\int_{S^1} f$ for the integral $\int_{-\pi}^{\pi} f(e^{i\theta})d\theta$.

Lemma 4.4. *Let ψ be a vector in the domain of $(L_0^\lambda)^2$ and let f be a real smooth function on S^1. Then the following hold:*

(i) $L^\lambda(f)\psi \in D(Q_\lambda)$ *and*

$$Q_\lambda L^\lambda(f)\psi = L^\lambda(f)Q_\lambda\psi + \frac{i}{2}G^\lambda(f')\psi.$$

(ii) $G^\lambda(f)\psi \in D(Q_\lambda)$ *and*

$$Q_\lambda G^\lambda(f)\psi = -G^\lambda(f)Q_\lambda\psi + 2L^\lambda(f)\psi - \frac{c}{24\pi}\left(\int_{S^1} f\right)\psi.$$

(iii) $G^\lambda(f)\psi \in D(G^\lambda(f))$ *and*

$$G^\lambda(f)^2\psi = L^\lambda(f^2)\psi + \frac{c}{12\pi}\left(\int_{S^1}(f'^2 - \frac{1}{4}f^2)\right)\psi.$$

Proof. These are rather straightforward consequences of the Ramond algebra relations in Eq. (6) and of the energy bounds in Eq. (9) and Eq. (10) together with the fact that V_λ is a core for every power of L_0^λ. $\square$

Proposition 4.5. *Let f be a real smooth function on S^1. If $\alpha \in \mathbb{R}$ and $|\alpha|$ is sufficiently large then, for every $\psi \in D((L_0^\lambda)^2)$, $(L^\lambda(f) + i\alpha)^{-1}\psi \in D((L_0^\lambda)^2)$ and*

$$Q_\lambda(L^\lambda(f) + i\alpha)^{-1}\psi = (L^\lambda(f) + i\alpha)^{-1}Q_\lambda\psi - \frac{i}{2}(L^\lambda(f) + i\alpha)^{-1}G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1}\psi.$$

Proof. By Proposition 4.3 if $|\alpha|$ sufficiently large then $(L^\lambda(f) + i\alpha)^{-1}\psi \in D((L_0^\lambda)^2)$, for any $\psi \in D((L_0^\lambda)^2)$. Hence, by Lemma 4.4 (i), $L^\lambda(f)(L^\lambda(f) + i\alpha)^{-1}\psi \in D(Q_\lambda)$ and

$$Q_\lambda L^\lambda(f)(L^\lambda(f) + i\alpha)^{-1}\psi = L^\lambda(f)Q_\lambda(L^\lambda(f) + i\alpha)^{-1}\psi + \frac{i}{2}G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1}\psi.$$

Adding $i\alpha Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi$ to both sides of the previous equality we find

$$Q_\lambda\psi = (L^\lambda(f)+i\alpha)Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi + \frac{i}{2}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}\psi,$$

so that

$$(L^\lambda(f)+i\alpha)Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi = Q_\lambda\psi - \frac{i}{2}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}\psi$$

and the conclusion follows by letting $(L^\lambda(f)+i\alpha)^{-1}$ act to both sides of the latter equality. □

Proposition 4.6. *Let f_1 and f_2 be real smooth functions on S^1 and assume that $f_1^2 \leq Cf_2$ for some $C>0$. Then, for any nonzero $\alpha \in \mathbb{R}$*

$$G^\lambda(f_1)(L^\lambda(f_2)+i\alpha)^{-1} \in B(\mathcal{H}_\lambda).$$

Proof. Let $\beta \in \mathbb{R}$. By Proposition 4.3, if $|\beta|$ is sufficiently large we have

$$(L^\lambda(f)+i\beta)^{-1}D((L_0^\lambda)^2) \subset D((L_0^\lambda)^2)$$

and consequently $G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}$ is densely defined. Moreover,

$$\left(G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}\right)^* \supset (L^\lambda(f_2)-i\beta)^{-1}G^\lambda(f_2)$$

is also densely defined and hence $G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}$ is closable.

From Lemma 4.4 *(iii)* it follows that

$$\begin{aligned}\|G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}\psi\|^2 &= ((L^\lambda(f_2)+i\beta)^{-1}\psi, L^\lambda(f_1^2)(L^\lambda(f_2)+i\beta)^{-1}\psi)\\ &+ \frac{c}{12\pi}\left(\int_{S^1}(f_1'^2-\frac{1}{4}f_1^2)\right)\|(L^\lambda(f_2)+i\beta)^{-1}\psi\|^2,\end{aligned}$$

for all $\psi \in D((L_0^\lambda)^2)$. By assumption $Cf_2 - f_1^2 \geq 0$ and hence, as a consequence of [8, Theorem 4.1], $L^\lambda(Cf_2-f_1^2)$ is bounded from below. It follows that there exists $\tilde{C}>0$ such that

$$\begin{aligned}\|G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}\psi\|^2 &\leq ((L^\lambda(f_2)+i\beta)^{-1}\psi, (CL^\lambda(f_2)+\tilde{C})(L^\lambda(f_2)+i\beta)^{-1}\psi)\\ &\leq \|(L^\lambda(f_2)-i\beta)^{-1}(CL^\lambda(f_2)+\tilde{C})(L^\lambda(f_2)+i\beta)^{-1}\|\cdot\|\psi\|^2\\ &\leq \left(\frac{C}{2|\beta|}+\frac{\tilde{C}}{|\beta|^2}\right)\|\psi\|^2,\end{aligned}$$

for all $\psi \in D((L_0^\lambda)^2)$. Therefore $G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}$ restricts to a bounded linear map on $D((L_0^\lambda)^2)$ and, since it is closable, it must be bounded on its domain. Moreover, since $(L^\lambda(f_2)+i\beta)^{-1}$ belongs to $B(\mathcal{H}_\lambda)$ and $G^\lambda(f_1)$ is closed, $G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}$ is closed. Accordingly $G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1} \in B(\mathcal{H}_\lambda)$. Now, if $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, the operator $(L^\lambda(f_2)+i\beta)(L^\lambda(f_2)+i\alpha)^{-1}$ belongs to $B(\mathcal{H}_\lambda)$. Hence

$$\begin{aligned}G^\lambda(f_1)(L^\lambda(f_2)+i\alpha)^{-1} &= G^\lambda(f_1)(L^\lambda(f_2)+i\beta)^{-1}(L^\lambda(f_2)+i\beta)(L^\lambda(f_2)+i\alpha)^{-1}\\ &\in B(\mathcal{H}_\lambda).\end{aligned}$$

□

Lemma 4.7. *Let f be a real smooth function on S^1 such that* $\mathrm{supp} f \subset \overline{I}$ *and $f(z) > 0$ for all $z \in I$, for some interval $I \in \mathcal{I}$. Assume moreover that $f'(z) \neq 0$ for all $z \in I$ sufficiently close to the boundary. Then there exists $C > 0$ such that $f'^2 \leq Cf$.*

Proof. Let h be the real function on S^1 defined by

$$h(z) \equiv \begin{cases} 0 & \text{if } z \in \overline{I'}, \\ \frac{f'^2(z)}{f(z)} & \text{if } z \in I. \end{cases}$$

Clearly h is continuous at every point of $I \cup I'$ and the restriction of h to I' is continuous. Now let ζ be a boundary point of I and let z_n be a sequence in I converging to ζ. Then, by L'Hospital's rule,

$$\begin{aligned} \lim_{n\to\infty} h(z_n) &= \lim_{n\to\infty} \frac{f'^2(z_n)}{f(z_n)} = \lim_{n\to\infty} \frac{2f'(z_n)f''(z_n)}{f'(z_n)} \\ &= 2f''(\zeta) = 0 = h(\zeta). \end{aligned}$$

It follows that h is continuous on S^1 and consequently it is bounded from above by some constant $C > 0$. Then $f'^2 = hf \leq Cf$. □

Theorem 4.8. *Let α be a real number and let f be as in Lemma 4.7. Then if $|\alpha|$ is sufficiently large, $(L^\lambda(f) + i\alpha)^{-1} \in D(\delta)$ and*

$$\delta\big((L^\lambda(f) + i\alpha)^{-1}\big) = -\frac{i}{2}(L^\lambda(f) + i\alpha)^{-1} G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1}.$$

Proof. Let α be any nonzero real number and let

$$b \equiv -\frac{i}{2}(L^\lambda(f) + i\alpha)^{-1} G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1}.$$

By Lemma 4.7 and Proposition 4.6, $G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1} \in B(\mathcal{H}_\lambda)$ and hence $b \in B(\mathcal{H}_\lambda)$. If $|\alpha|$ is sufficiently large then, by Proposition 4.5 and the fact that $(L^\lambda(f) + i\alpha)^{-1}$ is even (it commutes with Γ_λ), we have

$$Q_\lambda(L^\lambda(f) + i\alpha)^{-1}\psi = \gamma_s\big((L^\lambda(f) + i\alpha)^{-1}\big)Q_\lambda\psi + b\psi,$$

for all $\psi \in D((L_0^\lambda)^2)$ and since $D((L_0^\lambda)^2)$ is a core for Q_λ the conclusion follows from Lemma 2.4. □

Proposition 4.9. *Let f be a real smooth function on S^1. If $\alpha \in \mathbb{R}$ and $|\alpha|$ is sufficiently large then, for every $\psi \in D((L_0^\lambda)^2)$,*

$$(L^\lambda(f) + i\alpha)^{-1}\psi \in D((L_0^\lambda)^2),\ (L^\lambda(f) + i\alpha)^{-1}G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1}\psi \in D(L_0^\lambda),$$
$$G^\lambda(f)(L^\lambda(f) + i\alpha)^{-1}\psi \in D(Q_\lambda)$$

and

$$\begin{aligned} Q_\lambda G^\lambda(f)(L^\lambda(f) + i\alpha)^{-1}\psi &= -G^\lambda(g)(L^\lambda(f) + i\alpha)^{-1}Q_\lambda\psi \\ &+ \left(2L^\lambda(f) - \frac{c}{24\pi}\int_{S^1} f\right)(L^\lambda(f) + i\alpha)^{-1}\psi \\ &+ \frac{i}{2}G^\lambda(f)(L^\lambda(f) + i\alpha)^{-1}G^\lambda(f')(L^\lambda(f) + i\alpha)^{-1}\psi. \end{aligned}$$

Proof. It follows from Proposition 4.3 that if $|\alpha|$ is sufficiently large then

$$(L^\lambda(f)+i\alpha)^{-1}D((L_0^\lambda)^k) \in D((L_0^\lambda)^k), \text{ for } k=1,2.$$

Now let $\psi \in D((L_0^\lambda)^2)$ so that $(L^\lambda(f)+i\alpha)^{-1}\psi \in D((L_0^\lambda)^2)$. By Lemma 4.4 (ii), we have

$$\begin{aligned} Q_\lambda G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}\psi &= -G^\lambda(f)Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi \\ &+ \left(2L^\lambda(f) - \frac{c}{24\pi}\int_{S^1} f\right)(L^\lambda(f)+i\alpha)^{-1}\psi. \end{aligned}$$

Moreover, by Proposition 4.5 we have

$$Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi = (L^\lambda(f)+i\alpha)^{-1}Q_\lambda\psi - \frac{i}{2}(L^\lambda(f)+i\alpha)^{-1}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}\psi.$$

From the fact that $(L^\lambda(f)+i\alpha)^{-1}\psi \in D((L_0^\lambda)^2)$ and $Q_\lambda\psi \in D(L_0^\lambda)$ we have that $Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi$ and $(L^\lambda(f)+i\alpha)^{-1}Q_\lambda\psi$ belong to $D(L_0^\lambda)$. Hence

$$(L^\lambda(f)+i\alpha)^{-1}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}\psi$$

also belongs to $D(L_0^\lambda) \subset D(G^\lambda(f))$ and

$$\begin{aligned} G^\lambda(f)Q_\lambda(L^\lambda(f)+i\alpha)^{-1}\psi &= G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}Q_\lambda\psi \\ &- \frac{i}{2}G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}\psi. \end{aligned}$$

It follows that

$$\begin{aligned} Q_\lambda G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}\psi &= -G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}Q_\lambda\psi \\ &+ \frac{i}{2}G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}\psi \\ &+ \left(2L^\lambda(f) - \frac{c}{24\pi}\int_{S^1} f\right)(L^\lambda(f)+i\alpha)^{-1}\psi. \end{aligned}$$

□

Lemma 4.10. *Let α be a nonzero real number and let f be a real nonnegative smooth function on S^1. Then $G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1} \in B(\mathcal{H}_\lambda)$.*

Proof. Since f is continuous on S^1 then it is bounded from above by some constant $C>0$. Accordingly $f^2 \le Cf$ and the conclusion follows from Proposition 4.6. □

Theorem 4.11. *Let α be a real number and let f be as in Lemma 4.7. Then, if $|\alpha|$ is sufficiently large, $G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1} \in D(\delta)$ and*

$$\begin{aligned} \delta\big(G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}\big) &= \left(2L^\lambda(f) - \frac{c}{24\pi}\int_{S^1} f\right)(L^\lambda(f)+i\alpha)^{-1} \\ &+ \frac{i}{2}G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}. \end{aligned}$$

Proof. Let α be any nonzero real number. We denote $G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}$ by a and

$$\left(2L^\lambda(f) - \frac{c}{24\pi}\int_{S^1} f\right)(L^\lambda(f)+i\alpha)^{-1} + \frac{i}{2}G^\lambda(f)(L^\lambda(f)+i\alpha)^{-1}G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}$$

by b. From Lemma 4.10 we know that $a \in B(\mathcal{H}_\lambda)$ and it is easy to see that $\gamma_\lambda(a) = -a$. It is also evident that

$$\left(2L^\lambda(f) - \frac{c}{24\pi}\int_{S^1} f\right)(L^\lambda(f)+i\alpha)^{-1} \in B(\mathcal{H}_\lambda).$$

By Lemma 4.7 and Proposition 4.6 we also know that $G^\lambda(f')(L^\lambda(f)+i\alpha)^{-1}$ belongs to $B(\mathcal{H}_\lambda)$. As a consequence $b \in B(\mathcal{H}_\lambda)$. Now, if $|\alpha|$ is sufficiently large, it follows from Proposition 4.9, that $a\psi \in D(Q_\lambda)$ and

$$Q_\lambda a\psi = \gamma_\lambda(a)Q_\lambda\psi + b\psi,$$

for all $\psi \in D((L_0^\lambda)^2)$. The conclusion then follows from Lemma 2.4 because $D((L_0^\lambda)^2)$ is a core for Q_λ. □

Lemma 4.12. $\mathcal{A}_\lambda(I) \cap D(\delta)$ *is a strongly dense unital *-subalgebra of* $\mathcal{A}_\lambda(I)$ *for all* $I \in \mathcal{I}$.

Proof. By Proposition 2.5 $\mathcal{A}_\lambda(I) \cap D(\delta)$ is a unital *-subalgebra of $\mathcal{A}_\lambda(I)$ and hence, by von Neumann density theorem, it is enough to show that

$$(\mathcal{A}_\lambda(I) \cap D(\delta))' \subset \mathcal{A}_\lambda(I)'.$$

To this end let f be an arbitrary real smooth function on S^1 with support in I. Recalling that I must be open it is easy to see that there is an interval $I_0 \in \mathcal{I}$ such that $\overline{I_0} \subset I$ and supp$f \subset I_0$ and a smooth function g on S^1 such that supp$g \subset \overline{I_0}$, $g(z) > 0$ for all $z \in I_0$, $g'(z) \neq 0$ for all $z \in I_0$ sufficiently close to the boundary and $g(z) = 1$ for all $z \in$ supp$f$. Accordingly, there is a real number $s > 0$ such that $f(z) + sg(z) > 0$ for all $z \in I_0$. Now let $f_1 = f + sg$ and $f_2 = sg$. Then $f = f_1 - f_2$. Moreover f_1 and f_2 satisfy the assumptions in Lemma 4.7 and have support in I. Hence it follows from Theorem 4.8, Theorem 4.11 and the definition of $\mathcal{A}_\lambda(I)$ that there exists a nonzero real number α such that the operators $(L^\lambda(f_i)+i\alpha)^{-1}$ and $G^\lambda(f_i)(L^\lambda(f_i)+i\alpha)^{-1}$, $i = 1, 2$, belong to $(\mathcal{A}_\lambda(I) \cap D(\delta))$. As a consequence if $a \in (\mathcal{A}_\lambda(I) \cap D(\delta))'$. Then a commutes with $L^\lambda(f_i)$ and $G^\lambda(f_i)$, $i = 1, 2$. Therefore, if $\psi_1, \psi_2 \in C^\infty(L_0^\lambda)$ then,

$$\begin{aligned}
(a\psi_1, L^\lambda(f)\psi_2) &= (a\psi_1, L^\lambda(f_1)\psi_2) - (a\psi_1, L^\lambda(f_2)\psi_2) \\
&= (aL^\lambda(f_1)\psi_1, \psi_2) - (aL^\lambda(f_2)\psi_1, \psi_2) \\
&= (aL^\lambda(f)\psi_1, \psi_2)
\end{aligned}$$

and, since $C^\infty(L_0^\lambda)$ is a core for $L^\lambda(f)$, it follows that a commutes with $L^\lambda(f)$ and hence with $e^{iL^\lambda(f)}$. Similarly a commutes with $e^{iG^\lambda(f)}$. Hence $a \in \mathcal{A}_\lambda(I)'$ and the conclusion follows. □

Now we can state and prove the main result of this section.

Theorem 4.13. *$\mathfrak{A}_\lambda(I)$ is a strongly dense unital *-subalgebra of $\mathcal{A}_\lambda(I)$ for all $I \in \mathcal{I}$.*

Proof. Let $I_0 \in \mathcal{I}$ be an interval whose closure $\overline{I_0}$ is contained in I and let $a \in \mathcal{A}_\lambda(I_0) \cap D(\delta)$. Now, if the support of the function $f \in C_c^\infty(\mathbb{R})$ is sufficiently close to 0 then

$$a_f = \int_{\mathbb{R}} e^{itL_0^\lambda} a e^{-itL_0^\lambda} f(t)\mathrm{d}t \in \mathcal{A}_\lambda(I).$$

Moreover, by Proposition 2.12, $a_f \in C^\infty(\delta)$ and thus $a_f \in \mathfrak{A}_\lambda(I)$. It follows that $\mathcal{A}_\lambda(I_0) \cap D(\delta) \subset \mathfrak{A}_\lambda(I)''$ and by Lemma 4.12 that $\mathcal{A}_\lambda(I_0) \subset \mathfrak{A}_\lambda(I)''$ By conformal covariance we have

$$\mathcal{A}_\lambda(I) = \bigvee_{\overline{I_0} \subset I} \mathcal{A}_\lambda(I_0)$$

and hence $\mathcal{A}_\lambda(I) \subset \mathfrak{A}_\lambda(I)''$. □

We have thus proved the following.

Theorem 4.14. *Let λ be a unitary, graded, positive energy representation of the Ramond algebra and denote as above by L_n^λ, G_r^λ, $n, r \in \mathbb{Z}$, the Virasoro elements and the Fermi elements. Assume that $\mathrm{Tr}(e^{-\beta L_0^\lambda}) < \infty$ for all $\beta > 0$. Then, with $\mathcal{A}_\lambda$ the associated net of local von Neumann algebras on S^1, we have a net of graded, θ-summable spectral triples (in fact a net of quantum algebras) $(\mathfrak{A}_\lambda, \mathcal{H}_\lambda, Q_\lambda)$ where $Q_\lambda \equiv G_0^\lambda$ such that $\mathfrak{A}_\lambda(I)$ is a strongly dense unital *-subalgebra of $\mathcal{A}_\lambda(I)$ for every interval $I \subset S^1$ in $\mathcal{I}$.*

In particular this is the case if λ is the irreducible unitary representation with central charge c and lowest weight $h_\lambda = c/24$ (minimal lowest weight) and the Fredholm index is equal to 1.

Remark 4.15. If the graded unitary positive energy representation λ of the Ramond algebra is a direct of sum of finitely many irreducible (not necessarily graded) subrepresentations then $\mathrm{Tr}(e^{-\beta L_0^\lambda}) < \infty$ for all $\beta > 0$ and hence the above theorem applies. The same is true also for certain infinite direct sums of irreducibles.

Remark 4.16. The irreducible unitary representations of the Ramond algebra with lowest weight $h_\lambda \neq c/24$ are not graded. Nonetheless in this case the above Theorem 4.14 gives so-called *odd* spectral triples.

5 Spectral triples from the Neveu-Schwarz algebra

The construction in the preceding section can be adjusted to obtain spectral triples from representations of the Neveu-Schwarz algebra. The essential difficulty in this case is that, while the Ramond algebra contains the global supercharge operator $Q = G_0$, this is not true for the Neveu-Schwarz algebra: here one may define δ by abstract commutation relations, but then one is soon faced with the question whether this formal superderivation still has a nontrivial domain as in Sect. 2. We shall overcome this problem here below.

At this point we should make a comment. The nets of generalised quantum algebras that we shall construct provide an intrinsic structure visible in any representation, both

in the Ramond and in the Neveau-Schwarz case. Indeed if a given representation of the Neveu-Schwarz algebra is "locally normal" with respect to a reprensentation of the Ramond algebra with the same central charge, as it is natural to expect (but difficult to prove in general), then one could carry the generalised quantum algebra from one representation to the other one. It is however unclear to us that one can naturally associate a cyclic cocycle to any net of generalised quantum algebras. In this respect the interesting representations so far appear to be the representations of the Ramond algebra.

The *Neveu-Schwarz algebra* is the super-Lie algebra generated by even elements L_n, $n \in \mathbb{Z}$, odd elements G_r, $r \in \mathbb{Z} + 1/2$, and a central even element k, satisfying the relations (6). In other words the commutation relations of the Ramond algebra hold also here but the index r runs through $\mathbb{Z} + 1/2$.

We shall consider representations λ of the Neveu-Schwarz algebra by linear endomorphisms, denoted by $L^\lambda_m, G^\lambda_r, k^\lambda$, $m \in \mathbb{Z}$, $r \in \mathbb{Z} + 1/2$, of a complex vector space V_λ equipped with an involutive linear endomorphism Γ_λ inducing the super-Lie algebra grading. The endomorphisms $L^\lambda_m, G^\lambda_r, k^\lambda$, satisfy (6) with respect to the brackets given by the super-commutator induced by Γ_λ, and we suppose they satisfy the properties corresponding to $(i) - (iv)$ as stated for the Ramond algebra in the preceding section. Moreover we assume that λ is a positive energy representation, namely that $(v, L^\lambda_0 v) \geq 0$ for all $v \in V_\lambda$. (For the representations of the Ramond algebra positivity of the energy was a consequence of $(i) - (iv)$ but this is not the case for the Neveu-Schwarz algebra.) If λ is an irreducible unitary positive energy representation with lowest weight h_λ then, in contrast with the Ramond case, it is automatically graded by $\Gamma_\lambda = e^{i2\pi(L^\lambda_0 - h_\lambda}$ and in fact it satisfies all the above assumptions.

As in Section 4, the elements L^λ_m, G^λ_r define closable operators on the Hilbert space completion $\mathcal{H}_\lambda$ of V_λ and their closure is denoted by the same symbol. We also have the linear energy bounds (9) and (10). Note however, that, in contrast with the Ramond case, the unitary positive energy representations (not necessarily irreducible) are automatically graded by $\Gamma_\lambda = e^{i2\pi L^\lambda_0}$.

For the Fermi Neveu-Schwarz fields we shall consider a Fourier expansion with respect to a different basis as follows. Let f be a smooth function on S^1 with support contained in some interval $I \in \mathcal{I}_0$, where $\mathcal{I}_0$ is defined as in Section 3. This is equivalent to require that supp$f$ does not contain the point -1. The Fourier coefficients here are

$$\hat{f}_r = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ir\theta} \frac{\mathrm{d}\theta}{2\pi}, \quad r \in \mathbb{Z} + \frac{1}{2},$$

and they are rapidly decreasing (to this end it is crucial that the support of f does not contains -1). Then because of the linear energy bounds the map

$$V_\lambda \ni v \mapsto \sum_{r \in \mathbb{Z}+1/2} \hat{f}_r G^\lambda_r v$$

defines a closable operator $G^\lambda(f)$ on $\mathcal{H}_\lambda$ whose closure is denoted by $G^\lambda(f)$ again; $L^\lambda(f)$ is defined as in the preceding section. The domains of $G^\lambda(f)$ and $L^\lambda(f)$ contain $D(L^\lambda_0)$ and they leave invariant $C^\infty(L^\lambda_0)$. Moreover, if f is real, $L^\lambda(f)$ and $G^\lambda(f)$ are selfadjoint operators and their restriction to any core for L^λ_0 are essentially selfadjoint operators cf. [3]. Actually in the case of $L^\lambda(f)$ the above properties hold without any restriction on the support of $f \in C^\infty(S^1)$.

We can then define a net $\mathcal{A}_\lambda$ of von Neumann algebras on $S^1 \setminus \{-1\}$ by

$$\mathcal{A}_\lambda(I) \equiv \{e^{iL^\lambda(f)}, e^{iG^\lambda(f)} : f \in C^\infty(S^1) \text{ real, supp} f \subset I\}'', \ I \in \mathcal{I}_0. \tag{19}$$

As in [4, Sect. 6.3] it can be shown that $\mathcal{A}_\lambda$ extends to a graded-local conformal covariant net on S^1 (in general without vacuum vector). In fact, we have

$$e^{i4\pi L_0^\lambda} \in \bigcap_{I \in \mathcal{I}} \mathcal{A}_\lambda(I)' \tag{20}$$

so that the action of $\mathrm{Diff}^{(\infty)}(S^1)$ on the local algebras factors through $\mathrm{Diff}^{(2)}(S^1)$.

Using the definition of smeared fields and the (anti-) commutation relations of the Neveu-Schwarz algebra, we get:

Proposition 5.1. *Let f, g be smooth functions on S^1 with support in some $I \in \mathcal{I}_0$. Then the smeared fields of the Neveu-Schwarz algebra satisfy the following (anti-) commutation relations on the common invariant core $C^\infty(L_0^\lambda)$:*

$$\begin{aligned} [L^\lambda(f), L^\lambda(g)] &= -iL^\lambda(f'g) + iL^\lambda(fg') + i\frac{c}{24\pi}\int_{S^1}(f'''g + f'g), \\ [L^\lambda(f), G^\lambda(g)] &= iG^\lambda(fg') - \frac{i}{2}G^\lambda(f'g), \\ [G^\lambda(f), G^\lambda(g)] &= 2L^\lambda(fg) + \frac{c}{6\pi}\int_{S^1}(f'g' - \frac{1}{4}fg). \end{aligned} \tag{21}$$

Remark 5.2. The above relation also holds, without any restriction on the supports of the smooth functions f and g, when λ is a representation of the Ramond algebra as in Section 4.

Now let φ be any real smooth function on S^1 with support in some interval in $\mathcal{I}_0$. Then $G^\lambda(\varphi)$ is an odd self-adjoint operator on the graded Hilbert space $\mathcal{H}_\lambda$ and hence we can define as in Section 2 a corresponding superderivation $\delta_\varphi = [G^\lambda(\varphi), \cdot]$ on $B(\mathcal{H}_\lambda)$ which, by (i) and (ii) in Proposition 2.1 is odd and antisymmetric.

We now make the following observation: with φ any function from the subset $\mathcal{C}_I \subset C^\infty(S^1, \mathbb{R})$ defined by

$$\mathcal{C}_I \equiv \{\varphi \in C^\infty(S^1, \mathbb{R}) : \varphi(z) = 1 \ \forall z \in I, -1 \notin \mathrm{supp}\varphi\} \tag{22}$$

and f a smooth function with support in I, we obtain form Proposition 5.1 the relations $(i) - (ii)$ of Lemma 4.4 with Q_λ replaced by $G^\lambda(\varphi)$ (the precise domain statements follow again from the linear energy bounds). Therefore, for any $\varphi \in \mathcal{C}_I$, we may interpret $G(\varphi)$ as a *local supercharge* for $\mathcal{A}_\lambda(I)$. Actually, as a consequence of the following proposition we, we will be able to use these local supercharges to define a net of superderivations which has the desired commutation relations.

Proposition 5.3. *Let $\varphi, \tilde{\varphi} \in \mathcal{C}_I$, $I \in \mathcal{I}_0$, and let δ_φ and $\delta_{\tilde{\varphi}}$ the superderivations on $B(\mathcal{H}_\lambda)$ associated to the selfadjoint operators $G^\lambda(\varphi)$ and $G^\lambda(\tilde{\varphi})$ respectively. Then the following hold:*

(i) *$D(\delta_\varphi) \cap \mathcal{A}_\lambda(I) = D(\delta_{\tilde{\varphi}}) \cap \mathcal{A}_\lambda(I)$ and $\delta_\varphi(a) = \delta_{\tilde{\varphi}}(a)$ for all $a \in D(\delta_\varphi) \cap \mathcal{A}_\lambda(I)$.*

(ii) $\delta_\varphi(a) \in \mathcal{A}_\lambda(I)$ *for all* $a \in D(\delta_\varphi) \cap \mathcal{A}_\lambda(I)$.

Proof. (i) It is enough to show that if $a \in D(\delta_\varphi) \cap \mathcal{A}_\lambda(I)$ then $a \in D(\delta_{\tilde{\varphi}}) \cap \mathcal{A}_\lambda(I)$ and $\delta_{\tilde{\varphi}}(a) = \delta_\varphi(a)$. Given any $a \in D(\delta_\varphi) \cap \mathcal{A}_\lambda(I)$ and any $\psi_1 \in C^\infty(L_0^\lambda)$, we have $a\psi_1 \in D(G(\varphi))$ and

$$\begin{aligned}(a\psi_1, G^\lambda(\tilde{\varphi})\psi_2) &= (a\psi_1, G^\lambda(\varphi)\psi_2) + (a\psi_1, G^\lambda(\tilde{\varphi}-\varphi)\psi_2)\\ &= (G^\lambda(\varphi)a\psi_1, \psi_2) + (a\psi_1, G^\lambda(\tilde{\varphi}-\varphi)\psi_2),\end{aligned}$$

for all $\psi_2 \in C^\infty(L_0^\lambda)$. Now, from the fact that $\tilde{\varphi}-\varphi$ vanishes on I it follows that $G^\lambda(\tilde{\varphi}-\varphi)$ is affiliated with $\mathcal{A}_\lambda(I')$ and hence, by graded locality for the net $\mathcal{A}_\lambda$, it is also affiliated with $Z_\lambda \mathcal{A}_\lambda(I)' Z_\lambda^*$, where as before $Z_\lambda = (1 - i\Gamma_\lambda)/(1-i)$. Accordingly $a\psi_1 \in D(G^\lambda(\tilde{\varphi}-\varphi))$ and

$$G^\lambda(\tilde{\varphi}-\varphi)a\psi_1 = \gamma_\lambda(a)G^\lambda(\tilde{\varphi})\psi_1 - \gamma_\lambda(a)G^\lambda(\varphi)\psi_1$$

so that, recalling that $G^\lambda(\varphi-\tilde{\varphi})$ is selfadjoint, we have

$$(a\psi_1, G^\lambda(\tilde{\varphi})\psi_2) = (\delta_\varphi(a)\psi_1 + \gamma_\lambda(a)G^\lambda(\tilde{\varphi})\psi_1, \psi_2).$$

As $C^\infty(L_0^\lambda)$ is a core for $G^\lambda(\tilde{\varphi})$, it follows that $a\psi_1 \in D(G^\lambda(\tilde{\varphi}))$ and

$$G^\lambda(\tilde{\varphi})a\psi_1 = \delta_\varphi(a)\psi_1 + \gamma_\lambda(a)G^\lambda(\tilde{\varphi})\psi_1.$$

Since ψ_1 was an arbitrary vector in $C^\infty(L_0^\lambda)$ and the latter is a core for $G^\lambda(\tilde{\varphi})$, the conclusion follows using Lemma 2.4.

(ii) Let I_1 be any interval in $\mathcal{I}_0$ containing the closure $\overline{I}$ of I. By (i) we can assume that $\mathrm{supp}\varphi \subset I_1$. Then for any $b \in \mathcal{A}_\lambda(I_1)'$ and $\psi \in C^\infty(L_0^\lambda)$, we have

$$\begin{aligned}b\delta_\varphi(a)\psi =& bG^\lambda(\varphi)a\psi - b\gamma(a)G^\lambda(\varphi)\psi\\ =& G^\lambda(\varphi)ab\psi - \gamma(a)G^\lambda(\varphi)b\psi = \delta_\varphi(a)b\psi\end{aligned}$$

because $a \in \mathcal{A}_\lambda(I)$ and $G^\lambda(\varphi)$ is affiliated with $\mathcal{A}(I_1)$. So $\delta_\varphi(a) \in \mathcal{A}_\lambda(I_1)$. Since $I_1 \supset \overline{I}$ was arbitrary we obtain

$$\delta_\varphi(a) \in \bigcap_{I_1 \supset \overline{I}} \mathcal{A}_\lambda(I_1) = \mathcal{A}_\lambda(I),$$

where the last equality is a consequence of conformal covariance of the net $\mathcal{A}_\lambda$. $\square$

Now for for all $I \in \mathcal{I}_0$ and $\varphi \in \mathcal{C}_I$ we consider the unital *-algebra $\mathfrak{A}_\lambda(I) \equiv C^\infty(\delta_\varphi) \cap \mathcal{A}_\lambda(I)$ and the antisymmetric odd superderivation $\delta_I : \mathfrak{A}_\lambda(I) \mapsto \mathfrak{A}_\lambda(I)$ defined by $\delta_I \equiv \delta_\varphi|_{\mathfrak{A}_\lambda(I)}$, which do not depend on the choice of $\varphi \in \mathcal{C}_I$ and thus are well-defined. Accordingly, if $I_1, I_2 \in \mathcal{I}_0$, $I_1 \subset I_2$ and $\varphi \in \mathcal{C}_{I_2} \subset \mathcal{C}_{I_1}$ then

$$\begin{array}{rcl}\mathfrak{A}_\lambda(I_1) &=& C^\infty(\delta_\varphi) \cap \mathcal{A}_\lambda(I_1)\\ &\subset& C^\infty(\delta_\varphi) \cap \mathcal{A}_\lambda(I_2)\\ &=& \mathfrak{A}_\lambda(I_2).\end{array}$$

Moreover, $\delta_{I_2}|_{I_1} = \delta_{I_1}$. Therefore the map $\mathcal{I}_0 \ni I \mapsto \mathfrak{A}_\lambda$ defines a net of unital *-algebras on $S^1 \setminus \{-1\}$ and the map $\delta^\lambda : \mathcal{I}_0 \ni I \mapsto \delta_I$ is a net of σ-weakly closable antisymmetric odd superderivations of $\mathfrak{A}_\lambda$. Moreover, it can be shown that it extends to a net on the double cover $S^{1(2)}$.

Proposition 5.4. *Let $I \in \mathcal{I}_0$ and $a \in \mathfrak{A}_\lambda(I)$. Then, for every $\psi \in D(L_0^\lambda)$, $a\psi \in D(L_0^\lambda)$ and $L_0^\lambda a\psi - aL_0^\lambda\psi = \delta_I^2(a)\psi$.*

Proof. A closure argument shows that it is enough to prove the proposition for all $\psi \in C^\infty(L_0^\lambda)$. Let $\varphi \in \mathcal{C}_I$ and let $\delta_\varphi = [G^\lambda(\varphi), \cdot\,]$ be the corresponding superderivation on $B(\mathcal{H}_\lambda)$. By assumption $a \in D(\delta_\varphi^2)$ and hence, by (the proof of) Lemma 2.9, for any $\psi \in C^\infty(L_0^\lambda)$ we have $a\psi \in D(G^\lambda(\varphi)^2)$ and

$$G^\lambda(\varphi)^2 a\psi - aG^\lambda(\varphi)^2\psi = \delta_I^2(a)\psi.$$

Now, by Proposition 5.1 we have

$$G^\lambda(\varphi)^2\psi_1 = L^\lambda(\varphi^2)\psi_1 + \frac{c}{12\pi}\int_{S^1}(\varphi'^2 - \frac{1}{4}\varphi^2)\psi_1,$$

so

$$(a\psi, L^\lambda(\varphi^2)\psi_1) = (aL^\lambda(\varphi^2)\psi, \psi_1) + (\delta_I^2(a)\psi, \psi_1),$$

for all $\psi_1 \in C^\infty(L_0^\lambda)$. Thus, $a\psi$ is in the domain of $L^\lambda(\varphi^2)$ and

$$L^\lambda(\varphi^2)a\psi = aL(\varphi^2)\psi + \delta_I^2(a)\psi.$$

Now, for all $\psi_1 \in C^\infty(L_0^\lambda)$ we have

$$L^\lambda(\varphi^2)\psi_1 + L^\lambda(1-\varphi^2)\psi_1 = L_0^\lambda\psi_1$$

and since, as a consequence of the fact that $1-\varphi^2$ vanishes on I, $L^\lambda(1-\varphi^2)$ is an (even) operator affiliated with $\mathcal{A}_\lambda(I')$, we also know that $a\psi$ is in the domain of $L^\lambda(1-\varphi^2)$ and $L^\lambda(1-\varphi^2)a\psi = aL^\lambda(1-\varphi^2)\psi$. Accordingly

$$(a\psi, L_0^\lambda\psi_1) = (aL_0^\lambda\psi, \psi_1) + (\delta_I^2(a)\psi, \psi_1),$$

so that $a\psi \in D(L_0^\lambda)$ and $L_0^\lambda a\psi = aL_0^\lambda\psi + \delta_I^2(a)\psi$. □

As a consequence of Proposition 5.4 and of the discussion preceding it we can conclude that, provided that $e^{-\beta L_0^\lambda}$ is trace class for all $\beta > 0$, $(\mathfrak{A}_\lambda, \mathcal{H}_\lambda, \delta_\lambda)$ is a net of generalized quantum algebras on $S^1 \setminus \{-1\}$ with Hamiltonian L_0^λ as defined in Section 3. Yet we do not know whether the algebras $\mathfrak{A}_\lambda(I)$ are dense in $\mathcal{A}_\lambda(I)$ or nontrivial at all, so let us now consider this point. We fix any interval $I_0 \in \mathcal{I}_0$, any function $\varphi \in \mathcal{C}_{I_0}$ and consider the superderivation $\delta_\varphi = [G^\lambda(\varphi), \cdot\,]$ on $B(\mathcal{H}_\lambda)$ as above. We can apply the theory from Section 4 to this setting again. Then one checks that all the statements from Proposition 4.3 through Lemma 4.12 hold true if we replace Q_λ by $G^\lambda(\varphi)$, δ by δ_φ and consider only functions in $C^\infty(S^1)$ with support contained in I_0. In particular we have the following analogue of Lemma 4.12.

Lemma 5.5. *Let I_0 be any interval in $\mathcal{I}_0$ and let $\varphi \in \mathcal{C}_{I_0}$. Then $D(\delta_\varphi) \cap \mathcal{A}_\lambda(I)$ is a strongly dense unital *-subalgebra of $\mathcal{A}_\lambda(I)$ for every $I \in \mathcal{I}_0$ such that $\overline{I} \subset I_0$.*

As I_0 was arbitrary, this will lead to an analogue of Theorem 4.14, but first we need to adapt the essential ingredient from Section 2 to the present situation, namely Proposition 2.12. As in Section 4 for any $f \in C_c^\infty(\mathbb{R})$ and any $a \in B(\mathcal{H}_\lambda)$ we set

$$a_f \equiv \int_{\mathbb{R}} e^{itL_0^\lambda} a e^{-itL_0^\lambda} f(t)\mathrm{d}t. \tag{23}$$

Proposition 5.6. *Let I be any interval in $\mathcal{I}_0$ and let $\varphi \in \mathcal{C}_I$. Moreover let I_0 be any interval in $\mathcal{I}_0$ whose closure $\overline{I_0}$ is contained in I. Then there exists $\varepsilon > 0$ such that for all $f \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp} f \subset (-\varepsilon, \varepsilon)$ and all $a \in D(\delta_\varphi) \cap \mathcal{A}(I_0)$ we have $a_f \in \mathfrak{A}_\lambda(I)$.*

Proof. We basically work as in the preceding sections. However, here we have to take care to remain "local" in order to preserve the right commutation relations.

Since $\varphi \in \mathcal{C}_I$ its support is contained in some $I_1 \in \mathcal{I}_0$. Fix ε such that $e^{it}I_1 \in \mathcal{I}_0$ and $e^{it}I_0 \subset I$ for all $t \in (-\varepsilon, \varepsilon)$. Given $a \in D(\delta_\varphi) \cap \mathcal{A}_\lambda(I_0)$, by rotation covariance of the net $\mathcal{A}_\lambda$ we have

$$e^{itL_0^\lambda} a e^{-itL_0^\lambda} \in \mathcal{A}_\lambda(e^{it}I_0) \subset \mathcal{A}_\lambda(I),$$

for all $t \in (-\varepsilon, \varepsilon)$. Hence, if the support of the function $f \in C_c^\infty(\mathbb{R})$ is contained in $(-\varepsilon, \varepsilon)$ we also have $a_f \in \mathcal{A}_\lambda(I)$. Now from the definition of the smeared fields in the representation λ it easily follows that $e^{itL_0^\lambda} G^\lambda(\varphi) e^{-itL_0^\lambda} = G^\lambda(\varphi_t)$ for all $t \in (-\varepsilon, \varepsilon)$, where the function φ_t is defined by $\varphi_t(z) = \varphi(e^{-it}z)$. Accordingly, for any $t \in (-\varepsilon, \varepsilon)$, $e^{itL_0^\lambda} a e^{-itL_0^\lambda} \in D(\delta_{\varphi_t}) \cap \mathcal{A}_\lambda(e^{it}I_0)$ and

$$\delta_{\varphi_t}\big(e^{itL_0^\lambda} a e^{-itL_0^\lambda}\big) = e^{itL_0^\lambda} \delta_\varphi(a) e^{-itL_0^\lambda}.$$

Now for all $t \in (-\varepsilon, \varepsilon)$, we have $\varphi_t, \varphi \in \mathcal{C}_{e^{it}I_0}$ and hence by Proposition 5.3 we can conclude that $e^{itL_0^\lambda} a e^{-itL_0^\lambda}$ belongs to $D(\delta_\varphi)$ and that

$$\delta_\varphi\big(e^{itL_0^\lambda} a e^{-itL_0^\lambda}\big) = \delta_{\varphi_t}\big(e^{itL_0^\lambda} a e^{-itL_0^\lambda}\big)$$

so that

$$\delta_\varphi\big(e^{itL_0^\lambda} a e^{-itL_0^\lambda}\big) = e^{itL_0^\lambda} \delta_\varphi(a) a e^{-itL_0^\lambda}.$$

It follows that $a_f \in D(\delta_\varphi)$ and $\delta_\varphi(a_f) = \delta_\varphi(a)_f$.

Next, for any $\psi \in C^\infty(L_0^\lambda)$, we have $a_f\psi, \delta_\varphi(a)_f\psi \in C^\infty(L_0^\lambda)$. Hence $\delta_\varphi(a_f)\psi = \delta_\varphi(a)_f\psi \in D(G^\lambda(\varphi))$ and (cf. the proof of Proposition 5.4 and Lemma 2.11)

$$\begin{aligned}
G^\lambda(\varphi)\delta_\varphi(a_f)\psi &= G^\lambda(\varphi)^2 a_f\psi - G^\lambda(\varphi)\gamma(a_f)G^\lambda(\varphi)\psi \\
&= G^\lambda(\varphi)^2 a_f\psi - \delta_\varphi(\gamma(a_f))G^\lambda(\varphi)\psi - a_f G^\lambda(\varphi)^2\psi \\
&= L^\lambda(\varphi_I^2) a_f\psi - a_f L(\varphi^2)\psi - \delta_\varphi(\gamma(a_f))G^\lambda(\varphi)\psi \\
&= L_0^\lambda a_f\psi - a_f L_0^\lambda\psi - \delta_\varphi(\gamma(a_f))G^\lambda(\varphi)\psi \\
&= i a_{f'}\psi - \delta_\varphi(\gamma(a_f))G^\lambda(\varphi)\psi \\
&= i a_{f'}\psi + \gamma(\delta_\varphi(a_f))G^\lambda(\varphi)\psi.
\end{aligned}$$

Thus $a_f \in D(\delta_\varphi^2)$ and $\delta_\varphi^2(a_f) = i a_{f'}$ by Lemma 2.4 and the conclusion easily follows by induction. □

With the preceding modification of Proposition 2.12 the following final result is proved in the same manner as Theorem 4.13.

Theorem 5.7. *$\mathfrak{A}_\lambda(I)$ is a strongly dense unital *-subalgebra of $\mathcal{A}_\lambda(I)$ for all $I \in \mathcal{I}_0$.*

We can summarise the main results of this section in the following theorem.

Theorem 5.8. *Let λ be a unitary, positive energy representation of the Neveu-Schwarz algebra with* $\mathrm{Tr}(e^{-\beta L_0^\lambda}) < \infty$ *for all* $\beta > 0$*. Then, with* $\mathfrak{A}_\lambda(I)$ *and* δ_I *as above,* $I \in \mathcal{I}_0$*, the triple* $(\mathfrak{A}_\lambda, \mathcal{H}_\lambda, \delta_\lambda)$ *is a net of generalised quantum algebras on* $S^1 \smallsetminus \{-1\}$ *with Hamiltonian* L_0^λ*.*

In particular this applies if λ is any irreducible unitary lowest weight representation of the Neveu-Schwarz algebra.

Corollary 5.9. *With λ as in the above theorem,* $(\mathfrak{A}_\lambda, \mathcal{H}_\lambda, \delta_\lambda)$ *extends to a rotation covariant net of generalised quantum algebras on the double cover* $S^{1(2)}$ *of* S^1 *with Hamiltonian* L_0^λ*. It does not extends to a net on* S^1*.*

Proof. According to the proof of Proposition 5.6, we have local rotation covariance, namely if I and $e^{it}I$ belong to $\mathcal{I}_0$ for all $|t| < \varepsilon$ for some $\varepsilon > 0$, then

$$e^{itL_0^\lambda}\mathfrak{A}_\lambda(I)e^{-itL_0^\lambda} = \mathfrak{A}_\lambda(e^{it}I)$$

and

$$\delta_{e^{it}I} = \mathrm{Ad}e^{itL_0^\lambda} \circ \delta_I \circ \mathrm{Ad}e^{-itL_0^\lambda} \ .$$

Since

$$e^{i4\pi L_0^\lambda} \in \bigcap_{I\in\mathcal{I}} \mathcal{A}_\lambda(I)',$$

the above equation and the group property of $t \mapsto e^{itL_0^\lambda}$ allow to extend consistently $(\mathfrak{A}_\lambda, \mathcal{H}_\lambda, \delta_\lambda)$ to a rotation covariant net of generalised quantum algebras on $S^{1(2)}$.

Since

$$e^{-i2\pi L_0^\lambda}\Gamma_\lambda \in \bigcap_{I\in\mathcal{I}} \mathcal{A}_\lambda(I)'$$

in the Neveu-Schwarz case, we have

$$\mathrm{Ad}e^{i2\pi L_0^\lambda} \cdot \delta_I \cdot \mathrm{Ad}e^{-i2\pi L_0^\lambda} = \mathrm{Ad}\Gamma_\lambda \cdot \delta_I \cdot \mathrm{Ad}\Gamma_\lambda = \gamma_\lambda \cdot \delta_I \cdot \gamma_\lambda = -\delta_I \ ,$$

namely the derivation δ_I associated with an interval $I \in \mathcal{I}^{(2)}$ changes sign after a 2π-rotation, so it cannot give rise to a net of generalised quantum algebras on S^1. □

Remark 5.10. In the Ramond case we found no obstruction to define the net $(\mathfrak{A}_\lambda, \mathcal{H}_\lambda, Q_\lambda)$ on S^1. This is due to the fact that if λ is a representation of the Ramond algebras then the unitary operator $e^{i2\pi L_0^\lambda}$ commutes with all the local algebras and hence it does not implement the grading γ_λ.

6 Outlook

By the results in this paper, we have the basis for the analysis of the JLO cyclic cocycle and index theorems. One point to further describe is a "universal algebra" whose representations give rise to the spectral triples (in this paper we have worked on the representation space from the beginning). Furthermore, there are different models, e.g. the supersymmetric free field. This kind of issues and analysis will be the subject of subsequent work.

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